A Minty variational principle for set optimization

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A B S T R A C T

Extremal problems are studied involving an objective function with values in (order) complete lattices of sets generated by so-called set relations. Contrary to the popular paradigm in vector optimization, the solution concept for such problems, introduced by F. Heyde and A. Löhne, comprises the attainment of the infimum as well as a minimality property. The main result is a Minty type variational inequality for set optimization problems which provides a sufficient optimality condition under lower semicontinuity assumptions and a necessary condition under appropriate generalized convexity assumptions. The variational inequality is based on a new Dini directional derivative for set-valued functions which is defined in terms of a "lattice difference quotient." A residual operation in a lattice of sets replaces the inverse addition in linear spaces. Relationships to families of scalar problems are pointed out and used for proofs. The appearance of improper scalarizations poses a major difficulty which is dealt with by extending known scalar results such as Diewert's theorem to improper functions.

1. Introduction

Throughout the paper, let X and Z be two locally convex, topological linear spaces and $C \subseteq Z$ a convex cone with $0 \in C$. Moreover, $\mathcal{P}(Z)$ denotes the set of all subsets of Z including $\emptyset$. Let a function $f : X \to \mathcal{P}(Z)$ be given. The basic problem is

$$\text{minimize } f \text{ subject to } x \in X.$$

Motivated by duality for vector optimization, such set-valued optimization problems have been considered first by Corley [9,10] and Dinh The Luc [44]. They gained popularity after the appearance of [42]

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and [39–41] in which so-called set relations are investigated and used to define minimality concepts for sets.

However, the power set $\mathcal{P}(Z)$ is too large an object and lacks reasonable structure which can be exploited for optimization purposes. On the other hand, additional assumptions imposed on f often imply that the images of f belong to a relatively small subset of $\mathcal{P}(Z)$ which carries a richer algebraic and order structure. For example, C-convexity of f (see [6, Definition 1.1]) implies that the set $f(x) + C$ is convex for all $x \in X$. Therefore, appropriate subsets of $\mathcal{P}(Z)$ are used as image sets of set-valued functions, for example in [26, 27,43,51], and we will follow this approach. The main goal is to define new lower directional derivatives of Dini type for set-valued functions and provide necessary and sufficient conditions in terms of variational inequalities of Minty type to characterize solutions of set-valued minimization problems.

Two questions arise. First, what is understood by a solution of the above problem? Secondly, how can a directional derivative, in particular a difference quotient, be defined if the image set of the function is not a linear space? The answer to the first question is a new solution concept for set-valued optimization problems proposed by F. Heyde and A. Löhne [30,43]. This concept subsumes classical minimality notions borrowed from vector optimization as well as the infimum/supremum in complete lattices (which are usually not present in vector optimization). The answer to the second is provided by means of residuation operations in (order) complete lattices of sets which replace the inverse addition (the difference) in linear spaces. This approach has been proposed in [27,28].

Several notions of derivatives for set-valued functions have been introduced, compare e.g. [1,2,12,13, 18,34,37,46,52] to mention but a few. Apart from approaches relying on an embedding procedure into a linear space or approaches similar to those in [12,37,52], usually some kind of tangent cone to the graph of f at a point $(x, z) \in X \times Z$ with $z \in f(x)$ is defined to be the graph of the derivative. In this paper, we define a set-valued derivative using increments of function values where the difference is replaced by a residual operation and thus provides a substitute for the difference quotient in linear spaces. A "lattice limit" procedure then provides the desired derivative.

It turns out that the lattice concepts are appropriate and sufficient to formulate Minty type variational inequalities which yield the desired characterizations for the new type of solutions. Minty variational inequalities have been introduced in [45] as the problem of finding some $\bar{x} \in K$ such that

$$\forall y \in K: \quad \langle F(y), \bar{x} - y \rangle \leq 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, and $K \subseteq \mathbb{R}^n$ is a non-empty convex subset. This inequality proved to be useful to study primitive optimization problems when F is some derivative of the objective function $f : \mathbb{R}^n \to \mathbb{R}$. The main result in this field is known as Minty variational principle and basically states that the Minty variational inequality provides a sufficient optimality condition for minimizers of f under a lower semicontinuity assumption. The same inequality is also a necessary optimality condition under generalized convexity type assumptions. In [11], the Minty variational principle has been applied to a non-differentiable scalar optimization problem using lower Dini derivatives. The same approach has been extended to the vector case in [12].

The main purpose of this paper is to provide a Minty variational principle for set optimization problems. In the process we also need to deepen the study of lower semicontinuity and generalized convexity. Indeed, it turns out that known results on generalized convexity need to be extended to cover the case of improper functions, which is, to the best of our knowledge, not covered by the existing literature.

The paper is organized as follows. In Section 2, basic notation and results on the "lattice approach" to set optimization are introduced. The notion of a conlinear space as a natural setting for the image space of classes of set-valued functions is presented in Section 2.2. The solution concept for set optimization problems and scalarization techniques are described subsequently. In Section 3, the Dini-type derivative for set-valued functions is introduced, while in Section 4 generalized convexity concepts for possibly improper scalar and