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Intersection and proximity of processes of flats

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ABSTRACT

Weakly stationary random processes of k-dimensional affine subspaces (flats) in $\mathbb{R}^n$ are considered. If $2k \geq n$, then intersection processes are investigated, while in the complementary case $2k < n$ a proximity process is introduced. The intensity measures of these processes are described in terms of parameters of the underlying k-flat process. By a translation into geometric parameters of associated zonoids and by means of integral transformations, several new uniqueness and stability results for these processes of flats are derived. They rely on a combination of known and novel estimates for area measures of zonoids, which are also developed in the paper. Finally, an asymptotic second-order analysis as well as central and non-central limit theorems for length-power direction functionals of proximity processes derived from stationary Poisson k-flat process complement earlier works for intersection processes.

1. Introduction

The aim of this work is to discuss the interplay and duality of the notions 'intersection' and 'proximity' for locally finite random systems X of k-dimensional affine subspaces (called *k-flats*, for short) in n-dimensional Euclidean spaces $\mathbb{R}^n$, $n \geq 2$. More precisely, we consider weakly stationary point processes X on the space $A(n,k)$ of k-flats in $\mathbb{R}^n$ for $k \in \{0,\dots,n-1\}$. Here we call X weakly stationary if its intensity measure is translation invariant, see Section 2 for precise definitions. Stationary k-flat processes are one of the basic models in stochastic geometry and have been studied intensively, for example, in [2,13,20,28,29,33,37–39,41], see also [35] and the references therein. While stationarity implies weak stationarity, the converse is not true in general. However, for Poisson processes the two concepts are equivalent.

If $k \geq n/2$, any two k-flats of X, which are in general position, intersect in a $(2k-n)$-dimensional subspace. This gives rise to the *intersection process* $X_{(2)}$ of order two of X. Under a suitable condition on X, the process $X_{(2)}$ is again weakly stationary, its intensity is called the *intersection density* $\gamma_{(2)}(X)$ of X. Even if X is a Poisson process, $X_{(2)}$ is not a Poisson process any more. For a stationary Poisson

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process X, the intensity measure of the intersection process was described in [35, Theorem 4.4.9] and, in the case of a stationary Poisson hyperplane process X, an upper bound for the intersection density was given in [35, Theorem 4.6.5], based on the method of associated zonoids. Analogously, intersection processes $X_{(r)}$ of higher order $r \in \{2, \dots, n\}$ and their intensities $\gamma_{(r)}(X)$ were considered in [35, Theorem 4.4.8], where $X_{(r)}$ arises from the intersections of any selection of r hyperplanes in general position of a stationary Poisson hyperplane process X.

If X is a weakly stationary process of k-flats with $1 \le k < n/2$, a somehow dual situation arises. Again under a suitable condition on X, any pair (E, F) of disjoint k-flats of X in general position has now a positive distance $d(E, F) > 0$, which is attained in uniquely determined points $x_E \in E$ and $x_F \in F$. They give rise to a random line segment $s(E, F) = \overline{x_E x_F}$, the perpendicular of E and F. If we only consider pairs $E, F \in X$ with distance $d(E, F) \le \delta$, for a fixed distance threshold $\delta > 0$, then the segment-midpoints $m(E, F) = (x_E + x_F)/2$ build a weakly stationary point process in $\mathbb{R}^n$, the intensity of which is denoted by $\pi(X, \delta)$ (without the distance condition, the midpoints may even lie dense in $\mathbb{R}^n$). The quantity $\pi(X, \delta)$ is called *proximity* of X with distance threshold δ. For a stationary Poisson process X and $\delta = 1$, it has been introduced and studied by Schneider [33] (see also [35, Theorem 4.4.10]). It is known that if X is a stationary Poisson line process in $\mathbb{R}^n$ with fixed intensity, $\pi(X, \delta)$ attains its maximal value if and only if X is isotropic (compare [35, Theorem 4.6.6]).

The mentioned duality is based on the fact that stationary Poisson processes X of k-flats are in one-to-one correspondence with pairs $(\gamma, \mathbb{Q})$, where $\gamma > 0$ is the intensity and $\mathbb{Q}$ is the directional distribution of X, a probability measure on the Grassmannian $G(n, k)$ of k-dimensional linear subspaces of $\mathbb{R}^n$. This fact allows us to associate with X a dual Poisson process $X^\perp$ of $(n-k)$-flats, which has the same intensity γ and whose directional distribution $\mathbb{Q}^\perp$ is given by the image of $\mathbb{Q}$ under the orthogonal complement map $L \mapsto L^\perp$. Then, writing κ_{n-2k} for the volume of the $(n-2k)$-dimensional unit ball and $\pi(X)$ for $\pi(X, 1)$, we have the relation

$$\pi(X) = \kappa_{n-2k}\, \gamma_{(2)}\big(X^\perp\big)$$

according to the main result in [33] or the remark after Theorem 4.4.10 in [35] (both cited results contain a factor 1/2 which has to be removed). This clearly expresses the relation between the proximity $\pi(X)$ and the second intersection density $\gamma_{(2)}(X^\perp)$.

In this paper, we consider the *proximity process* built by the segments $s(E, F)$, described above, and we study intersection processes and proximity processes as well as their interplay in more detail and in greater generality. In particular, we work in the framework of weakly stationary k-flat processes. This is also the setting which allows us to construct examples (in Appendix A) of k-flat processes which are not Poisson, but have moment properties similar to that of Poisson processes. To enhance the readability of our text and to make the paper self-contained, Section 2 contains the basic notions and results from convex and stochastic geometry which are needed in the following. In particular, we recall the notion of an associated zonoid and provide a brief description of mixed volumes and area measures. In Section 3, we prove a stability result for the area measures of zonoids which cannot be found in the existing literature. Then, in Section 4, we discuss intersection processes and present some extensions and generalizations of results in [35], in particular a stability estimate for the directional distribution of a flat process X if the directional distribution of a lower-dimensional intersection process is given. In Section 5, we introduce the proximity process Φ associated with a weakly stationary k_1-flat process X_1 and a weakly stationary k_2-flat process X_2, if X_1 and X_2 are stochastically independent and $k_1 + k_2 < n$. This is the random process of all line segments $s(E, F)$ perpendicular to flats $E \in X_1$ and $F \in X_2$ in general position and such that the length of $s(E, F)$ is bounded from above by a prescribed distance threshold. We describe its intensity measure in terms of the intensities and the directional distributions of the original processes X_1 and X_2. We also consider the proximity process Φ of a single k-flat process X with $1 \le k < n/2$. For a Poisson line