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# On approximate isometries and application to stability of a functional equation

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ABSTRACT

In this paper, we show a generalized stability of isometries between Banach spaces. Making use of this result, we prove the corresponding stability of the functional equation $\|f(x-y)\| = \|f(x) - f(y)\|$.



## 1. Introduction

Let $E$ and $F$ be Banach spaces, and let $f$ be a mapping from $E$ to $F$. Put

$$\varepsilon = \sup\big\{\big|\|f(x) - f(y)\| - \|x - y\|\big| : x, y \in E\big\}.$$

The $f$ is called an $\varepsilon$-isometry, if $\varepsilon < \infty$. In 1945, Hyers and Ulam [17] first studied $\varepsilon$-isometries, and they asked whether any surjective $\varepsilon$-isometry is close to an isometry. This question is well-known as the Hyers–Ulam problem, which is also called stability problem of isometries. In 1983, making use of a result of Gruber [14], Gevirtz [13] solved the problem. Indeed, he showed that for any surjective $\varepsilon$-isometry $f : E \to F$ there exists an isometry $U : E \to F$ so that

$$\big\|f(x) - U(x)\big\| \le 5\varepsilon. \tag{1.1}$$

The constant 5 in (1.1) has been improved to 2 (see [24,29]).

Although the surjective assumption in the Hyers–Ulam problem cannot be removed, it can be relaxed. We recall that $f : E \to F$ is $\delta$-surjective, if for every $y \in F$ there exists $x \in E$ with $\|f(x) - y\| \le \delta$ (see [3]).

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Dilworth [9] and Tabor [33] studied $\delta$-surjective $\varepsilon$-isometries independently. The optimal approximation result for $\delta$-surjective $\varepsilon$-isometries can be found in [30]. For general nonsurjective $\varepsilon$-isometries, see [7,8,26,30]. The $\varepsilon$-isometries in bounded sets of Euclidean spaces have also been studied (see [1,35,36]).

In 1985, Lindenstrauss and Szankowski [22] studied a wider concept of approximate isometries. For a surjective mapping $f$ from $E$ to $F$, they introduced the function

$$\varphi_f(t) = \sup\{\big|\|f(x) - f(y)\| - \|x - y\|\big| : \|x - y\| \le t \text{ or } \|f(x) - f(y)\| \le t\}$$

for $t \ge 0$. They showed that if

$$\int_1^\infty \frac{\varphi_f(t)}{t^2} dt < \infty,$$

then there is an isometry $U$ from $E$ onto $F$ so that

$$\|f(x) - U(x)\| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

Recently, Dong [11] showed that this result is also valid under $\delta$-surjective assumption instead of surjective assumption.

In 2000, motivated by the result of Lindenstrauss and Szankowski [22], Dolinar [10] showed the following theorem.

**Theorem 1.1** *(Dolinar). Let $0 \le p < 1$, $\varepsilon > 0$ and let $f : E \to F$ be a surjective mapping with $f(0) = 0$. If*

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \le \varepsilon \|x - y\|^p$$

*for all $x, y \in E$, then there exists an isometry $U : E \to F$ such that*

$$\|f(x) - U(x)\| \le \varepsilon K(p)\|x\|^p \quad \text{for all } x \in E,$$

*where $K(p)$ is a constant independent of $E$ and $F$.*

The question which arises naturally in view of the result of Dilworth [9] is whether Theorem 1.1 is also valid under $\delta$-surjective assumption. In Section 2, we investigate this question by modifying the error $|\|f(x) - f(y)\| - \|x - y\||$ when $\|x - y\|$ is small.

Related to the stability problem of isometries, the stability problem of functional equations was also posed by Ulam [34]. The first partial solution to this question was given by Hyers [16]. In 1978, Rassias [27] showed the following celebrated theorem, which generalized the result of Hyers.

**Theorem 1.2** *(Rassias). Let $f$ be a map from a Banach space $E$ into a Banach space $F$, and assume that*

$$\|f(x + y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

*for some $\theta > 0$, $0 \le p < 1$, and for all $x, y \in E$. Then there is a unique additive map $T : E \to F$ which satisfies*

$$\|f(x) - Tx\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

*for all $x \in E$.*