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Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains

Yavar Kian [a,b], Quang Sang Phan [c], Eric Soccorsi [a,b,*]

[a] *Aix-Marseille Université, CNRS, CPT UMR 7332, 13288 Marseille, France*
[b] *Université de Toulon, CNRS, CPT UMR 7332, 83957 La Garde, France*
[c] *Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, 30-348 Kraków, Poland*

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A B S T R A C T

We consider the inverse problem of determining the time independent scalar potential of the dynamic Schrödinger equation in an infinite cylindrical domain from one boundary Neumann observation of the solution. We prove Hölder stability by choosing the Dirichlet boundary condition suitably.

Contents

* Corresponding author.
E-mail addresses: yavar.kian@univ-amu.fr (Y. Kian), Quang.Sang.Phan@im.uj.edu.pl (Q.S. Phan), eric.soccorsi@univ-amu.fr (E. Soccorsi).

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1. Statement of the problem and results

We continue our analysis of the inverse problem of determining the scalar potential $q : \Omega \to \mathbb{R}$ in an unbounded quantum cylindrical domain $\Omega = \omega \times \mathbb{R}$, where ω is a connected bounded open subset of $\mathbb{R}^{n-1}$, $n \geq 2$, with no less than C^2-boundary $\partial\omega$, from partial Neumann data. This may be equivalently reformulated as to whether the electrostatic disorder occurring in Ω, modelling an idealized straight carbon nanotube, can be retrieved from the partial boundary observation of the quantum wave propagating in Ω. We refer to [7, §1.2] for the discussion on the physical motivations and the relevance of this model. Namely we seek stability in the identification of q from partial Neumann measurement of the solution u to the following initial boundary value problem

$$\begin{cases} -iu' - \Delta u + qu = 0, & \text{in } Q := (0,T) \times \Omega \\ u(0,x) = u_0(x), & x \in \Omega \\ u(t,x) = g(t,x), & (t,x) \in \Sigma := (0,T) \times \Gamma. \end{cases} \tag{1.1}$$

Here $T > 0$ is fixed, $\Gamma := \partial\omega \times \mathbb{R}$ and $\partial/\partial t$ is denoted by $'$. Since Γ is unbounded we make the boundary condition in the last line of (1.1) more explicit. Writing $x := (x', x_n)$ with $x' := (x_1, \ldots, x_{n-1}) \in \omega$ for every $x \in \Omega$ we extend the mapping

$$C_0^\infty\big((0,T) \times \mathbb{R}; H^2(\omega)\big) \to L^2\big((0,T) \times \mathbb{R}; H^{3/2}(\partial\omega)\big)$$
$$v \mapsto \big[(t,x_n) \in (0,T) \times \mathbb{R} \mapsto v(t,\cdot,x_n)_{|\partial\omega}\big],$$

to a bounded operator from $L^2((0,T) \times \mathbb{R}; H^2(\omega))$ into $L^2((0,T) \times \mathbb{R}; H^{3/2}(\partial\omega))$, denoted by γ_0. Then for every $u \in C^0([0,T]; H^2(\Omega))$ the above mentioned boundary condition reads $\gamma_0 u = g$.

Throughout the entire paper we choose

$$g := \gamma_0 G_0, \quad \text{with } G_0(t,x) := u_0(x) + it(\Delta - q_0)u_0(x),\ (t,x) \in Q, \tag{1.2}$$

where $q_0 = q_0(x)$ is a given scalar function we shall make precise below.

In the particular case where q is *a priori* known outside some given compact subset of Ω and on the boundary Γ, it is shown in [7] that the scalar potential may be Lipschitz stably retrieved from one partial Neumann observation of the solution to (1.1) for suitable initial and boundary conditions u_0 and g. This result is similar to [2, Theorem 1], which was derived by Baudouin and Puel for the same operator but acting in a bounded domain. The main technical assumption common to [2,7] is that

$$u \in C^1\big([0,T]; L^\infty(\Omega)\big). \tag{1.3}$$

In this paper we pursue two main goals. First we want to analyze the direct problem associated to (1.1)–(1.2) in order to exhibit sufficient conditions on q and u_0 ensuring (1.3). Second, we aim to weaken the compactness condition imposed in [7] on the support of the unknown part of q, in the inverse problem of determining the scalar potential appearing in (1.1) from one partial Neumann observation of u.

The following result solves the direct problem associated to (1.1)–(1.2). Here and in the remaining part of this text we note $\|w\|_{j,\mathcal{O}}$, $j \in \mathbb{N}$, for the usual H^j-norm of w in any subset $\mathcal{O}$ of $\mathbb{R}^m$, $m \in \mathbb{N}^* := \{1, 2, \ldots\}$, where $H^0(\mathcal{O})$ stands for $L^2(\mathcal{O})$.

Theorem 1.1. *Let $k \geq 2$, assume that $\partial\omega$ is C^{2k} and pick*

$$(q_0, u_0) \in \big(W^{2k,\infty}(\Omega) \cap C^{2(k-1)}(\overline{\Omega}; \mathbb{R})\big) \times H^{2(k+1)}(\Omega),$$