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Ellipse-preserving Hermite interpolation and subdivision

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ABSTRACT

We introduce a family of piecewise-exponential functions that have the Hermite interpolation property. Our design is motivated by the search for an effective scheme for the joint interpolation of points and associated tangents on a curve with the ability to perfectly reproduce ellipses. We prove that the proposed Hermite functions form a Riesz basis and that they reproduce prescribed exponential polynomials. We present a method based on Green's functions to unravel their multi-resolution and approximation-theoretic properties. Finally, we derive the corresponding vector and scalar subdivision schemes, which lend themselves to a fast implementation. The proposed vector scheme is interpolatory and level-dependent, but its asymptotic behavior is the same as the classical cubic Hermite spline algorithm. The same convergence properties—i.e., fourth order of approximation—are hence ensured.

1. Introduction

Cubic Hermite splines are piecewise-cubic polynomial functions that are parametrized in terms of the value of the function and its derivative at the end point of each polynomial segment. By construction, the resulting spline is continuous with continuous first-order derivative. Cubic Hermite splines are used extensively in computer graphics and geometric modeling to represent curves as well as motion trajectories that pass through specified anchor points with prescribed tangents [13]. This is typically achieved by fitting a separate Hermite spline interpolant for each coordinate variable.

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Cubic Hermite splines have a number of attractive computational features. The basis functions are interpolating with a fourth-order approximation and their support is minimal. They satisfy multiresolution properties, which is the key to the specification of subdivision schemes [24] and the construction of multi-wavelet bases [9,27]. They are also closely linked to the Bézier curves, which provide an equivalent mode of representation. Their only limitation is that they require many control points to accurately reproduce elementary shapes such as circles and ellipses. This is why we investigate in this paper a variation of the classical Hermite scheme that is specifically geared towards the reproduction of elliptical shapes. This new Hermite subdivision scheme is obtained by studying the multiresolution properties of exponential Hermite splines. In particular, we deal with Hermite splines piecewisely spanned by linear polynomials, sine and cosine, often called cycloidal splines (see, e.g., [1,17]), which are ideally suited for outlining roundish objects in images by means of few control points (see [30] for an application of this spline model to the segmentation of biomedical images). Our main point in this work will be to show that we are able to achieve perfect ellipse reproduction while retaining all the attractive properties of the cubic Hermite splines modulo some proper adjustment of the underlying computational machinery. The extended Hermite functions that we shall specify are splines with pieces in $\mathcal{E}_4 := \langle 1, x, \mathrm{e}^{i\omega_0 x}, \mathrm{e}^{-i\omega_0 x}\rangle$, $\omega_0 \in [0, \pi]$, joining C^1-continuously at the integer knots. Hence they belong to the class of cycloidal splines. This points towards a connection with other exponential spline basis functions investigated in the literature (see, e.g., [10,18–22] and references quoted therein), although we are not aware of any prior work that specifically addresses the problem of ellipse reproduction nor covers the theoretical results that we are reporting here.

The paper is organized as follows. In Section 2, we motivate our design while spelling out the conditions that the basis functions must satisfy. We then derive the two Hermite functions $(\phi_{1,\omega_0}, \phi_{2,\omega_0})$ that fulfill our requirements in Section 3; these are the generators for the space $S^1_{\mathcal{E}_4}(\mathbb{Z})$, which is made up of functions that are piecewise exponential polynomials with (double) knots on the integers. In Section 4, we make the connection with exponential splines explicit by expressing the generators in terms of Green's functions of the differential operators $\mathrm{L}_{1,\omega_0} := \frac{\mathrm{d}^4}{\mathrm{d}x^4} + \omega_0^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and $\mathrm{L}_{2,\omega_0} := \frac{\mathrm{d}^3}{\mathrm{d}x^3} + \omega_0^2 \frac{\mathrm{d}}{\mathrm{d}x}$ (whose corresponding E-spline spaces are denoted as $S_{\mathcal{E}_4}(\mathbb{Z})$ and $S_{\mathcal{E}_3}(\mathbb{Z})$). In Section 5, we prove that the integer translates form a Riesz basis by analyzing the corresponding Gramian matrix. Section 6 is devoted to the characterization of the Hermite representation on $h\mathbb{Z}$, while Section 7 focuses on the investigation of its multi-resolution properties and the derivation of the corresponding subdivision scheme. In Section 8, we show that our cycloidal Hermite splines, in direct analogy with their polynomial counterpart, admit a Bézier representation that involves an exponential generalization of the classical Bernstein polynomials. Finally, in Section 9, we exploit the Bézier connection to derive the exponential version of the four point scalar subdivision scheme for the classical Hermite splines [15,29].

2. Motivation for the construction

An active contour (a.k.a. snake) is a computational tool for detecting and outlining objects in digital images. Its central component is a closed parametric curve that evolves spatially towards the contour of a target by minimizing a suitable energy functional [23]. The most commonly-used curve models rely on B-spline basis functions [2].

Since roundish objects are common place in biological imaging (in particular, fluorescence microscopy), it is of interest to develop a parametric framework that is specifically tailored to this type of shape while retaining the flexibility of splines and the ability to reparametrize by increasing the number of control points. A first solution to this problem was proposed by Delgado et al. who developed an "active cells" framework that is based on cardinal exponential B-splines [11]. The present research was motivated by the desire to refine this model by providing additional control over the tangents of the curve. This led us to the definition of a new parametric model that has the ability to perfectly reproducing ellipses while offering full tangential control as well as easy manipulation via the use of M control points and Bézier handles. By