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# Existence, computability and stability for solutions of the diffusion equation with general piecewise constant argument

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ABSTRACT

We study the solution of a class of PDE with piecewise constant argument of generalized type. Separation of variables leads to a solution formed by a series of products. In previous works, the convergence and bounds of the solution could be obtained from the study of the solution on the first constancy interval only. In the general case however, each term of the series may be unbounded at every interval, implying that the solution is not computable. We establish conditions where the convergence of the solution can be verified computing a finite number of terms of the series in each constancy interval, without requiring any regularity on the initial condition. Moreover, we combine asymptotic properties for each variable of the equation to obtain an exponential bound for the solution.


## 1. Introduction

Functional differential equations with deviated argument provide a mathematical model for systems where the changes of state depend upon its past history or its future. In [16], Myshkis proposed to study delayed differential equations with piecewise constant argument: DEPCA. These equations appear as an attempt to extend the theory of functional differential equations to systems with discontinuous argument deviations. DEPCA also arises in the process of replacing some terms of a differential equation by their piecewise constant approximations. This point of view has applications in impulsive or loaded differential equations of control theory, and stabilization of systems with discrete (sample) control [11,14,20,23,19]. A typical DEPCA is of the form

$$x'(t) = f\big(t, x(t), x\big(\gamma(t)\big)\big), \tag{1.1}$$

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where the argument $\gamma(t)$ has intervals of constancy. For example, delayed equations with $\gamma(t) = [t], [t-n]$, $t - n[t]$, advanced equations with $\gamma(t) = [t+n]$, and alternately retarded and advanced equations with $\gamma(t) = m[\frac{t+k}{m}]$ were investigated in [12,21,25] respectively, where $n, k, m \in \mathbb{N}$, $[\cdot]$ denotes the lowest-integer (floor) function, and $0 < k < m$.

Akhmet introduced in [1] the piecewise constant argument of generalized type, and the study of DEPCA with general deviation, so-called DEPCAG. Let $\{\theta_i\}$, $\{\zeta_i\}$ be real sequences, $I_i = [\theta_i, \theta_{i+1})$ and $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(t) = \zeta_i$, where $t \in I_i$, and $i \in \mathbb{N}_0$. The piecewise constant argument of generalized type assumes that

$$\begin{aligned}
&\theta_0 = 0 < \theta_i \le \zeta_i \le \theta_{i+1}, \qquad \lim_{i\to\infty} \theta_i \to \infty, \quad \text{and} \\
&\text{there exist} \quad 0 < l^* \le L^* < \infty \quad \text{such that} \quad l^* \le \theta_{i+1} - \theta_i \le L^*, \quad i \in \mathbb{N}.
\end{aligned} \tag{1.2}$$

For each $i \in \mathbb{N}_0$, $l_i = \theta_{i+1} - \theta_i$ represents the length of the interval of constancy, and $\zeta_i$ represents the deviation in $I_i$. The cases $\zeta_i = \theta_i, \theta_{i+1}$ correspond to the delayed and advanced cases respectively, and the case $\theta_i < \zeta_i < \theta_{i+1}$ corresponds to a general alternated advanced and retarded case.

Analytical results in DEPCA and DEPCAG focus on the existence, oscillation, and bounds of solutions [3,17,8,18,7,15,28,29,6]. These results have been applied in models of physical and artificial systems [5,14,31,9]. Moreover, the stability of numerical solutions of ordinary differential equations approximated by DEPCAG has received attention recently [10,30,22]. A detailed introduction to DEPCAG is in [4,5,25].

Partial differential equations with piecewise constant argument (PDEPCA) were introduced in [26] to study the existence, oscillation, and asymptotic bounds of the solutions of initial value problems with piecewise constant delays. For example, the wave equation with piecewise constant delays, boundary value problems, and initial value problems assuming piecewise constant arguments were investigated in [27,13,28] respectively. Complete descriptions for DEPCA and PDEPCA are in [23,24].

In this article we introduce the PDEPCA of generalized type, or PDEPCAG. Our aim is to extend the study of DEPCAG to partial differential systems. We deal with the following equation:

$$\begin{aligned}
&u_t(x,t) = a^2(t)u_{xx}(x,t) - b(t)u\big(x, \gamma(t)\big), \\
&u(0,0) = u(1,0) = 0, \\
&u(x,0) = u_0(x),
\end{aligned} \tag{1.3}$$

where $u_0$ is a continuous function on $[0,1]$, $a(t)$ and $b(t)$ are locally integrable functions on $[0,\infty)$ and $\gamma(t)$ is a step function as in (1.2). Eq. (1.3) describes the heat flow in a rod with a diffusion term $a^2(t)u_{xx}(x,t)$ along the rod, with gain or loss across lateral sides of the rod, measured at discrete times $-b(t)u(x,\gamma(t))$. An important example is given by the term $-b(t)u(x,t-2)$, which is simplified considering a step function $\gamma(t)$ near of $t-2$ [10]. In [13], Eq. (1.3) is studied considering two different piecewise constant arguments $\gamma(t) = [t]$ and $\gamma(t) = [t + \frac{1}{2}]$, with both $a(t) = a$ and $b(t) = b$ constant functions.

**Definition 1.** $u(x,t)$ is a *solution* of PDEPCAG (1.3) if $u(x,t)$ is continuous in $[0,1] \times [0,\infty)$, the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous in $[0,1] \times [0,\infty)$, with the possible exception of points $(x, \theta_n)$ where one sided partial derivatives exist, and $u(x,t)$ satisfies Eq. (1.3) with the possible exception of points $(x,\theta_n)$ $(n = 1, 2, \ldots)$.

Applying separation of variables on a PDEPCA leads to a solution defined by a series of products. In PDEPCA, the existence and bounds of the solution can be obtained from the study of the solution on the first constancy interval only [13,23,26,28]. This is because the length and deviation of the non-general piecewise constant argument assume the same values on each constancy interval. In particular, if a term