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Smoothness dependent stability in corrosion detection

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A R T I C L E I N F O

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A B S T R A C T

We consider the stability issue for the determination of a linear corrosion in a conductor by a single electrostatic measurement. We established a global *log–log* type stability when the corroded boundary is simply Lipschitz. We also improve such a result obtaining a global *log* stability by assuming that the damaged boundary is $C^{1,1}$-smooth.

1. Introduction

In this paper we study the stable determination of a corrosion coefficient on an inaccessible boundary by means of electrostatic measurements. Indeed, we adopt the model proposed by Inglese and Santosa, see [16] and the references therein (see also [10]), where the corrosion is represented by a non-negative exchange coefficient γ in a third-kind boundary condition. Such an adopted model can be regarded as a first linear approximation of a more accurate and nonlinear exchange condition discussed by Vogelius and others [12,21,24].

More precisely, we consider

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g, & \text{on } \Gamma_A, \\ \dfrac{\partial u}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma_I, \end{cases} \tag{1.1}$$

where Γ_A and Γ_I are two open, disjoint portions of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_A} \cup \overline{\Gamma_I}$ and $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. The portion Γ_A corresponds to the part of boundary which is accessible to measurements while Γ_I is the portion which is out of reach and where the corrosion damage occurs. The function $\gamma(x)$ is known as

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corrosion coefficient and its amplitude is related to the corrosion rate at the point x. The inverse problem we address here consists in the determination of such γ by means of the current density g prescribed on Γ_A and the corresponding measured potential $u|_{\Gamma_A}$. In particular, we are interested in providing *global* stability estimates for γ, or namely avoiding the a priori hypothesis that the unknown corrosion coefficient is a small perturbation of a given and known one.

Our first aim is to investigate the continuous dependence of γ upon the data when the corroded boundary Γ_I is merely *Lipschitz*. To this purpose, we notice that by the impedance condition in (1.1) we can formally compute γ as

$$\gamma(x) = -\frac{1}{u(x)}\frac{\partial u(x)}{\partial \nu}. \tag{1.2}$$

Since the potential u may vanish in some points on Γ_I, it follows that the above quotient may be highly unstable. In this respect it is necessary to compute the local vanishing rate of u on Γ_I. Indeed, we proved that such a rate can be controlled in an exponential manner as follows

$$\int\limits_{\Delta_r(x_0)} u^2 \geqslant \exp\bigl(-Kr^{-K}\bigr) \tag{1.3}$$

where $K > 0$ and $\Delta_r(x_0) = B_r(x_0) \cap \Gamma_I$ with $x_0 \in \Gamma_I^{2r} \subset \Gamma_I$ (see Section 2.1 for a precise definition) for sufficiently small radius r (see Section 4.1). By combining such a control with a logarithmic stability estimate for the underlying Cauchy problem we are able to prove a *global* stability estimate for γ with a *log–log* type modulus of continuity.

The second purpose of this paper is to strengthen the hypothesis on the corroded boundary assuming that Γ_I is $C^{1,1}$*-smooth* in order to obtain a better rate of stability. Indeed, under such additional a priori hypothesis, we derive a surface doubling inequality of this sort for sufficiently small radius r (see Section 4.2).

$$\int\limits_{\Delta_{2r}(x_0)} u^2 \leqslant const. \int\limits_{\Delta_r(x_0)} u^2, \tag{1.4}$$

which allows us to deduce that the vanishing rate of u at the boundary is at most polynomial, that is

$$\int\limits_{\Delta_r(x_0)} u^2 \geqslant \frac{1}{K} r^K, \tag{1.5}$$

for sufficiently small radius r (see Section 4.2). Again, by gathering a logarithmic stability estimate for the Cauchy problem and the above vanishing rate we provide a *global* stability estimate for γ with a *single log*.

In addition we also give an alternative proof of the above mentioned *global logarithmic* stability estimate. Such an alternative argument mostly relies on the application of the theory of the Muckenhoupt weights which justifies the computation in (1.2) in the $L^{\frac{2}{p-1}}$ sense for some $p > 1$.

Indeed, such a dependence of the modulus of continuity upon the smoothness of the boundary has been already observed in other contexts. In [3], inverse problems for the determination of unknown defects with Dirichlet and Neumann condition have been studied. The authors proved that when the unknown boundary is smooth enough and hence a doubling inequality at the boundary is available then stability turns out to be of *logarithmic* type. On the contrary relaxing the regularity assumptions on the unknown domain the rate of stability degenerates into a *log–log* type one.

Let us mention that global stability estimates for unknown boundary impedance coefficients have been previously discussed under analogous boundary smoothness assumptions in [8] and [22] for an inverse acoustic scattering problem.