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Complex zero strip decreasing operators

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ABSTRACT

Let $\phi(z)$ be a function in the Laguerre–Pólya class. Write $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ where $\alpha \geq 0$ and where $\phi_1(z)$ is a real entire function of genus 0 or 1. Let $f(z)$ be any real entire function of the form $f(z) = e^{-\gamma z^2} f_1(z)$ where $\gamma \geq 0$ and $f_1(z)$ is a real entire function of genus 0 or 1 having all of its zeros in the strip $S(r) = \{z \in \mathbb{C}: -r \leq \operatorname{Im} z \leq r\}$, where $r > 0$. If $\alpha\gamma < 1/4$, the linear differential operator $\phi(D)f(z)$, where D denotes differentiation, is known to converge to a real entire function whose zeros also belong the strip $S(r)$. We describe several necessary and sufficient conditions on $\phi(z)$ such that all zeros of $\phi(D)f(z)$ belong to a smaller strip $S(r_1) = \{z \in \mathbb{C}: -r_1 \leq \operatorname{Im} z \leq r_1\}$ where $0 \leq r_1 < r$ and r_1 depends on $\phi(z)$ but is independent of $f(z)$. We call a linear operator having this property a *complex zero strip decreasing operator* or CZSDO. We examine several relevant examples, in certain cases we give explicit upper and lower bounds for r', and we state several conjectures and open problems regarding complex zero strip decreasing operators.

1. Introduction

An important problem in the theory of the distribution of zeros of a collection of entire functions is to understand the effect of linear operators that act on the collection. It is particularly interesting when the operators preserve a nice property about the location of the zeros. The linear operators we will study in this paper are differential operators $\phi(D)$ where $\phi(z)$ is a function in the Laguerre–Pólya class and D is differentiation. If $f(z)$ is a real entire function satisfying appropriate technical requirements whose zeros belong to the strip $S(r) = \{z \in \mathbb{C}: -r \leq \operatorname{Im} z \leq r\}$, we study the problem of when all zeros of $\phi(D)f(z)$ belong to a smaller strip $S(r')$ where $0 \leq r' < r$. The main results in the paper are stated in Theorems 1.5 and 1.6.

Before stating these theorems we will need a few definitions and a technical lemma that defines the linear differential operator $\phi(D)$ and tells us when the expression $\phi(D)f(z)$ makes sense.

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Definition 1.1 *($\mathcal{LP}$ and $\mathcal{LP}_1$).* The *Laguerre–Pólya class*, denoted $\mathcal{LP}$, consists of the real entire functions whose Weierstrass product representations are of the form

$$cz^m e^{\alpha z-\beta z^2}\prod_k\left(1-\frac{z}{\alpha_k}\right)e^{z/\alpha_k}, \tag{1}$$

where c, α, β, α_k are real, $\beta \geq 0$, m is a nonnegative integer, and $\sum_k |\alpha_k|^{-2} < \infty$. The subclass $\mathcal{LP}_1$ of $\mathcal{LP}$ consists of those functions in $\mathcal{LP}$ with $\beta = 0$ in Eq. (1).

The class $\mathcal{LP}$ consists of the entire functions obtained as uniform limits on compact sets of sequences of real polynomials having only real zeros. See Levin [13, Thm. 3, p. 331]. Motivation for why this class of functions naturally arises in relation to differential operators is given in Section 2.

Definition 1.2 *($\mathcal{LP}(r)$ and $\mathcal{LP}_1(r)$).* For $r \geq 0$, the *extended Laguerre–Pólya class*, denoted $\mathcal{LP}(r)$, consists of the real entire functions having the Weierstrass product representation in Eq. (1) except that the zeros belong to the strip

$$S(r) = \{z \in \mathbb{C}\colon -r \leq \operatorname{Im} z \leq r\}.$$

Thus, the zeros of a function $f(z) \in \mathcal{LP}(r)$ are either real or occur in complex conjugate pairs. The subclass $\mathcal{LP}_1(r)$ of $\mathcal{LP}(r)$ consists of those functions in $\mathcal{LP}_1(r)$ with $\beta = 0$ in Eq. (1). If $r < 0$ or r is imaginary, we define $\mathcal{LP}(r) = \mathcal{LP}$ and $S(r) = \mathbb{R}$.

The following lemma shows how functions in $\mathcal{LP}$ define linear differential operators on functions in $\mathcal{LP}(r)$. A trivial modification to the proof of a theorem in Levin [13] gives:

Lemma 1.3. *(See Levin [13], Thm. 8, p. 360.) Assume*

$$\phi(z) = e^{-\gamma_1 z^2}\phi_1(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{LP}$$

where $\gamma_1 \geq 0$ and $\phi_1(z) \in \mathcal{LP}_1$. Also let $r \geq 0$ and assume $f(z) = e^{-\gamma_2 z^2} f_1(z) \in \mathcal{LP}(r)$ where $\gamma_2 \geq 0$ and $f_1(z) \in \mathcal{LP}_1(r)$. If $\gamma_1\gamma_2 < 1/4$, the linear differential operator $\phi(D)$ is defined by

$$\phi(D)f(z) = \sum_{k=0}^{\infty} a_k f^{(k)}(z), \tag{2}$$

where D denotes differentiation. The sum converges uniformly on every compact subset of $\mathbb{C}$ and $\phi(D)f(z) \in \mathcal{LP}(r)$.

The assumption $\gamma_1\gamma_2 < 1/4$ is essential. Levin [13, p. 361] gives the explicit example $\phi(z) = e^{-\gamma_1 z^2}$ and $f(z) = e^{-\gamma_2 z^2}$ to show that $\phi(D)f(z)$ diverges at $z = 0$ when $\gamma_1\gamma_2 = 1/4$.

In the lemma the zeros of $f(z)$ are in the strip $S(r)$ as are the zeros of $\phi(D)f(z)$. So, $\phi(D)$ is an operator that preserves the strip $S(r)$ containing the zeros. However, our main interest in this paper is to study the operators $\phi(D)$ such that the zeros of $\phi(D)f(z)$ belong to a strictly smaller strip $S(r_1)$ where $0 \leq r_1 < r$.