J. Math. Anal. Appl. 426 (2015) 423–439

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Killing magnetic curves in three-dimensional almost paracontact manifolds ☆

Giovanni Calvaruso [a,*], Marian Ioan Munteanu [b], Antonella Perrone [a]

[a] Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, Prov. Lecce-Arnesano, 73100, Lecce, Italy

[b] Department of Mathematics, University Alexandru Ioan Cuza Iasi, Bd. Carol I, n. 11, 700506 Iasi, Romania

ARTICLE INFO

Article history:
Received 25 October 2014
Available online 28 January 2015
Submitted by W. Sarlet

Keywords:
Almost paracontact metric structures
Normal structures
Killing vector fields
Magnetic curves

ABSTRACT

For an arbitrary three-dimensional normal paracontact metric structure equipped with a Killing characteristic vector field, we obtain a complete classification of the magnetic curves of the corresponding magnetic field. In particular, this yields to a complete description of magnetic curves for the characteristic vector field of three-dimensional paraSasakian and paracosymplectic manifolds. Explicit examples are described for the hyperbolic Heisenberg group and a paracosymplectic model.

1. Introduction

The magnetic curves on a pseudo-Riemannian manifold (M,g) are trajectories of charged particles, moving on M under the action of a magnetic field F. A *magnetic field* on (M,g) is a closed 2-form F on M, to which one may associate a skew-symmetric $(1,1)$-tensor field ϕ on M, called the Lorentz force, uniquely determined by $g(\phi X,Y) = F(X,Y)$, for any vector fields X, Y on M.

The *magnetic trajectories* of F are curves on M that satisfy the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = \phi(\dot\gamma)$. As such, they are a natural generalization of geodesics of M, that satisfy $\nabla_{\dot\gamma}\dot\gamma = 0$, which may be seen as the Lorentz equation in the absence of any magnetic field. However, it is relevant to note that the magnetic curves never reduce to geodesics. In fact, given a nontrivial magnetic field on a Riemannian manifold, there exists no linear connection, whose geodesics coincide with the magnetic curves of F [2, Prop. 2.1].

☆ First author partially supported by funds of the University of Salento and MIUR (within PRIN 2010-11). Second author partially supported by CNCS-UEFISCDI grant PN-II-RU-TE-2011-3-0017 (Romania). Third author partially supported by funds of the University of Salento. He wishes to thank the Department of Mathematics and Physics "E. de Giorgi" of the University of Salento for the warm hospitality during his visit in Spring 2014.

* Corresponding author.

E-mail addresses: giovanni.calvaruso@unisalento.it (G. Calvaruso), marian.ioan.munteanu@gmail.com (M.I. Munteanu), antonella_perrone@unisalento.it (A. Perrone).

http://dx.doi.org/10.1016/j.jmaa.2015.01.057
0022-247X/

In recent years, magnetic curves have been intensively studied (see for example [1–4,6,8,10] and the references therein), since they are a natural object of investigation under several points of view. In fact, besides their interpretation as a generalization of geodesics, they also encode further geometrical and physical meanings. Under a geometric point of view, magnetic curves are often related to slant curves with respect to some natural Killing vector fields. Moreover, they show a behavior similar to the one of classical helices (curves of constant curvature and torsion), and permit to characterize Lancret curves in the Euclidean space. Under the physical point of view, magnetic curves shape the magnetic flow in a background magnetic field, and describe the motion of charged particles in several physical scenarios (for example, in bending magnets in particle accelerators).

In this context, the three-dimensional case has been particularly investigated, because it shows some peculiar behaviors. Observe that on a three-dimensional pseudo-Riemannian manifold (M, g), the Lorentz force is provided via the cross product, and magnetic fields are in a one-to-one correspondence with divergence-free vector fields. In the case of a three-manifold of constant curvature, the Killing magnetic flow permits to clarify the relationships among phenomena of different physical meaning, codified by the Lorentz force equation for Killing magnetic fields, the field equation for Kirchhoff elastic rods (so giving a variational interpretation of magnetic curves), the Betchov–Da Rios equation for vortex filaments and the cubic Schröedinger equation [1].

In nonquantum relativity, the Lorentz world-force equation describes the dynamics of nonquantum relativistic particles in a four-dimensional spacetime. The connection between this approach and the classical Lorentz force equation on a Riemannian three-manifold is known (see for example [11]).

Among magnetic fields defined by divergence-free vector fields, a natural choice is to consider *Killing magnetic fields*, namely, the ones corresponding to Killing vector fields, and investigate magnetic curves on a three-dimensional manifold, related to some Killing vector fields which appear naturally in the geometry of the manifold itself.

The contact magnetic field determined by the characteristic vector field of any Sasakian three-manifold was studied in [3], determining the normal magnetic trajectories and proving that they are helices with axis the characteristic vector field itself. In dimension three, paracontact structures are the Lorentzian counterpart to contact Riemannian structures. It is then natural to consider the magnetic field corresponding to the Killing characteristic vector field ξ of an arbitrary normal paracontact metric three-manifold. Such a study is the purpose of the present paper, where we shall obtain a complete classification of the magnetic curves of ξ. We explicitly observe that the class of normal paracontact metric manifolds is very large. In particular, it includes paraSasakian and paracosymplectic manifolds. Moreover, with respect to the normal contact case, the study here is both more complex and interesting, because a metric compatible with an almost paracontact three-structure is Lorentzian and so, the vector $\nabla_{\dot\gamma}\dot\gamma$ can have any causal character.

The paper is organized in the following way. In Section 2 we shall report some basic information about three-dimensional almost paracontact metric structures and magnetic curves. Then, in Section 3 we shall describe magnetic curves of the characteristic vector field of normal paracontact metric structures, proving that they are helices (either space-like, time-like or light-like). As an application, explicit descriptions are given in Section 4 for the magnetic curves of a left-invariant paraSasakian structure of the hyperbolic Heisenberg group, and in Section 5 for magnetic curves of a paracosymplectic three-manifold.

2. Preliminaries

2.1. Almost paracontact metric structures

An *almost paracontact structure* on a $(2n + 1)$-dimensional (connected) smooth manifold M is a triple (φ, ξ, η), where φ is a $(1, 1)$-tensor, ξ a global vector field and η a 1-form, such that