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Axially symmetric incompressible MHD in three dimensions

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ABSTRACT

In this paper, we study the property of solutions to axially symmetric incompressible MHD equations in three dimensions. Firstly, we present the three-dimensional axially symmetric incompressible MHD equations. We propose a new one-dimensional model that approximates the MHD equations along the symmetric axis. To our surprise, this one-dimensional model can construct a family of exact solutions to the three-dimensional MHD equations. Secondly, we give a family of particular solutions to the three-dimensional and the prior estimates of one-dimensional MHD equations. Finally, we construct a family of global smooth solutions to the three-dimensional MHD equations by applying the one-dimensional solutions.

1. Introduction

Magnetohydrodynamics (MHD) [1] (magneto fluid dynamics or hydromagnetics) is the study of dynamics of electrically conducting fluids. Such fluids include plasmas, liquid metals, salt water and electrolytes. The word magnetohydrodynamic is derived from magneto-meaning magnetic field, hydro-meaning liquid and dynamics meaning movement. The field of MHD was initiated by Hannes Alfvén, for which he received the Nobel Prize in Physics in 1970.

The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid which in turn creates forces on the fluid and also changes the magnetic field itself. The set of equations which describe MHD are a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism.

This paper concerns itself with the incompressible three-dimensional MHD:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mu\Delta\mathbf{u} + (\nabla\times\mathbf{B})\times\mathbf{B}, \\ \mathbf{B}_t = \nu\Delta\mathbf{B} + \nabla\times(\mathbf{u}\times\mathbf{B}), \\ \nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{B} = 0. \end{cases} \tag{1.1}$$

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Here, $\mathbf{B}(\mathbf{x},t)$ denotes the magnetic field, $\mathbf{u}(\mathbf{x},t)$ is the velocity field, $p(\mathbf{x},t)$ is the pressure, $\mu \geq 0$ is the fluid viscosity, $\nu \geq 0$ is the resistivity. Firstly, by using the transformation of cylindrical coordinate, we obtain the three-dimensional axially symmetric incompressible MHD equation: (2.13) and (2.14). Secondly, we get a new one-dimensional model and this model approximates the three-dimensional asymmetric MHD along the symmetry axis. By expanding the angular velocity u^θ, the angular vorticity ω^θ, the angular stream function ψ^θ, the angular magnetic B^θ, the angular current density j^θ and the angular magnetic stream function Φ^θ around $r = 0$, we neglect the high-order terms in r and assume that the second partial derivative of u_1^θ, ω_1^θ, ψ_1^θ, B_1^θ, j_1^θ and Φ_1^θ with respect to z is much larger than the functions with respect to r. We get the one-dimensional coupled nonlinear partial differential equations are

$$(u_1^\theta)_t - 2(\psi_1^\theta)_z u_1^\theta + 2\psi_1^\theta(u_1^\theta)_z = (u_1^\theta)_{zz} - 2(\Phi_1^\theta)_z B_1^\theta + 2\Phi_1^\theta(B_1^\theta)_z, \tag{1.2}$$

$$(\omega_1^\theta)_t + 2\psi_1^\theta(\omega_1^\theta)_z - \left[(u_1^\theta)^2\right]_z = (\omega_1^\theta)_{zz} + 2\Phi_1^\theta(j_1^\theta)_z - \left[(B_1^\theta)^2\right]_z, \tag{1.3}$$

$$-(\psi_1^\theta)_{zz} = \omega_1^\theta, \tag{1.4}$$

$$(B_1^\theta)_t + 2\psi_1^\theta(B_1^\theta)_z = (B_1^\theta)_{zz} + 2\Phi_1^\theta(u_1^\theta)_z, \tag{1.5}$$

$$(j_1^\theta)_t - (\psi_1^\theta)_z j_1^\theta + 2\psi_1^\theta(j_1^\theta)_z = (j_1^\theta)_{zz} - (\Phi_1^\theta)_z\omega_1^\theta + 2\Phi_1^\theta(\omega_1^\theta)_z + 3(\psi_1^\theta)_z(\Phi_1^\theta)_{zz} - 3(\Phi_1^\theta)_z(\psi_1^\theta)_{zz}, \tag{1.6}$$

$$-(\Phi_1^\theta)_{zz} = j_1^\theta, \tag{1.7}$$

where $u_1^\theta = u_r^\theta(0,z,t)$ which is similar to ω_1^θ, ψ_1^θ, B_1^θ, j_1^θ and Φ_1^θ. Thirdly, we construct a family of exact solutions from the above one-dimensional model. If $(u_1^\theta, \omega_1^\theta, \psi_1^\theta, B_1^\theta, j_1^\theta, \Phi_1^\theta)$ is an exact solution to one-dimensional model, then

$$\begin{aligned} &u^\theta(r,z,t) = ru_1^\theta(z,t), \qquad \omega^\theta(r,z,t) = r\omega_1^\theta(z,t), \qquad \psi^\theta(r,z,t) = r\psi_1^\theta(z,t), \\ &B^\theta(r,z,t) = rB_1^\theta(z,t), \qquad j^\theta(r,z,t) = rj_1^\theta(z,t), \qquad \Phi^\theta(r,z,t) = r\Phi_1^\theta(z,t) \end{aligned} \tag{1.8}$$

is an exact solution to the three-dimensional axisymmetric MHD equations (Theorem 2.1). Fourthly, considering a family of particular solutions to three-dimensional axisymmetric MHD equations $u^\theta = 0$ and $B^r = B^z = 0$, we arrive at

$$\begin{cases} \bar\omega_t + 2\bar\psi(\bar\omega)_z = (\bar\omega)_{zz} - \left[(\bar B)^2\right]_z, \\ \bar B_t + 2\bar\psi(\bar B)_z = (\bar B)_{zz}, \\ \bar v_t + 2\bar\psi(\bar v)_z = (\bar v)_{zz} - \bar v^2 - \bar B^2 + e(t), \\ -(\bar\psi)_{zz} = \bar\omega, \end{cases} \tag{1.9}$$

where $\bar\omega = \omega_1^\theta$, $\bar\psi = \psi_1^\theta$, $\bar v = -(\bar\psi)_z = -(\psi_1^\theta)_z$, $\bar B = B_1^\theta$ and $e(t) = 3\int_0^1 \bar v^2 dz + \int_0^1 \bar B^2 dz$. Moreover, we get the prior estimates of $\bar B(z,t)$ and $\bar v(z,t)$ (Theorems 3.2–3.4). Finally, we construct a family of globally smooth solutions to the three-dimensional MHD by using the particular solutions to the one-dimensional model (1.9), then

$$\begin{aligned} &\tilde u^\theta(r,z,t) = r\left[\bar u_1(z,t)\phi(r) + u_1(r,z,t)\right], \qquad \tilde\omega^\theta(r,z,t) = r\left[\bar\omega_1(z,t)\phi(r) + \omega_1(r,z,t)\right], \\ &\tilde\psi^\theta(r,z,t) = r\left[\bar\psi_1(z,t)\phi(r) + \psi_1(r,z,t)\right], \qquad \tilde B^\theta(r,z,t) = r\left[\bar B_1(z,t)\phi(r) + B_1(r,z,t)\right], \\ &\tilde j^\theta(r,z,t) = r\left[\bar j_1(z,t)\phi(r) + j_1(r,z,t)\right], \qquad \tilde\Phi^\theta(r,z,t) = r\left[\bar\Phi_1(z,t)\phi(r) + \Phi_1(r,z,t)\right], \end{aligned}$$

where $(\bar u_1, \bar\omega_1, \bar\psi_1, \bar B_1, \bar j_1, \bar\Phi_1)$ is a particular solution to (1.2)–(1.7). We can find function $u_1(r,z,t)$, $\omega_1(r,z,t)$, $\psi_1(r,z,t)$, $B_1(r,z,t)$, $j_1(r,z,t)$ and $\phi_1(r,z,t)$ such that $\tilde u^\theta(r,z,t)$, $\tilde\omega^\theta(r,z,t)$, $\tilde\psi^\theta(r,z,t)$, $\tilde B^\theta(r,z,t)$, $\tilde j^\theta(r,z,t)$