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A priori estimates and existence of positive solutions for higher-order elliptic equations

Hatem Hajlaoui *, Abdellaziz Harrabi

Institut Supérieur des Mathématiques Appliquées et de l'Informatique de Kairouan, Université de Kairouan, Tunisia

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ABSTRACT

In this paper, we present a new sufficient condition to get a priori L^∞-estimates for positive solutions of higher-order elliptic equations in a smooth bounded convex domain of $\mathbb{R}^N$ with Navier boundary conditions or for radially symmetric solutions in the ball with Dirichlet boundary conditions. A priori L^∞-estimates for positive solutions of the second-order elliptic system in a smooth bounded convex domain of $\mathbb{R}^N$ with Dirichlet boundary conditions are also established. As usual, these a priori bounds allow us to obtain existence results. Also, by truncation technique combined with minimax method, we obtain existence of positive solution for higher-order elliptic equations of the form (1.1) below when we only assume that the nonlinearity is a nondecreasing positive function satisfying: $\liminf_{s\to+\infty} \frac{f(s)}{s} > \Lambda_1$, $\limsup_{s\to 0} \frac{f(s)}{s} < \Lambda_1$, where Λ_1 is the first eigenvalue of $(-\Delta)^m$ with Navier boundary conditions and the weak subcritical growth condition: $\lim_{s\to+\infty} \frac{f(s)}{s^\sigma} = 0$, where $\sigma = \frac{N+2m}{N-2m}$.

1. Introduction

In this paper, we are concerned with the existence of positive solutions for the following polyharmonic equations with Navier boundary conditions:

$$\begin{cases} (-\Delta)^m u = f(u) \quad \text{in } \Omega, \\ u = \Delta u = \cdots = (\Delta)^{m-1} u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded convex domain of $\mathbb{R}^N$, $N \geq 2$, and f is a nonlinearity to be specified later; or under Dirichlet boundary conditions in the ball B_R of $\mathbb{R}^N$, $N \geq 2$, centred at the origin with radius R:

* Corresponding author.
E-mail addresses: hajlouihatem@gmail.com (H. Hajlaoui), abdellaziz.harrabi@yahoo.fr (A. Harrabi).

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$$\begin{cases} (-\Delta)^m u = f(u) & \text{in } B_R, \\ u = Du = \cdots = (D)^{m-1}u = 0 & \text{on } \partial B_R. \end{cases} \tag{1.2}$$

In the last decades, problems related to existence and nonexistence of positive solutions for higher-order elliptic equations and systems have received a good deal of attention; see [4,9,10,13,15,16,12]. When variational methods cannot be employed, the question of existence of solutions may be dealt via topological methods. Using known topological fixed-point theorems, the proof of existence is essentially reduced to deriving a priori estimates for all possible solutions which are obtained in general by bootstrap or blow-up arguments [4,10,16,15,14].

When f has an asymptotic behaviour like s^p at infinity (with $1 < p < \frac{N+2m}{N-2m}$), Soranzo [16] established existence results for Eqs. (1.1) and (1.2) by combining L^∞-estimates with topological degree theory, which gave a direct extension of the results for $m = 1$ in [5]. The methods used in [16] to obtain L^∞ bounds are based on the well-known Pohozaev identity and boot-strap argument (in the Navier problem) and blow-up argument (in the Dirichlet problem in the ball). However, we can remark that under assumptions of Theorem 4 in [16], the energy functional associated with (1.1) satisfies the Palais–Smale condition and has the geometry of Mountain Pass lemma, consequently one can deduce the existence of a solution to (1.1) having the minimax structure. Moreover, since $f'(s) \geq 0$ with $f(0) = 0$ implies that $f(s) \geq 0$, the maximum principle ensures that such a solution is positive.

Let us fix notation: Consider the following Hilbert spaces:

$$H^m_\vartheta(\Omega) := \left\{ v \in H^m(\Omega);\ \Delta^j v = 0 \text{ on } \partial\Omega,\ \text{for } j < \frac{m}{2} \right\},$$
$$H^m_0(\Omega) := \left\{ v \in H^m(\Omega);\ D^j u = 0 \text{ on } \partial\Omega,\ \text{for } j = 0, 1, \ldots, m-1 \right\},$$

which are endowed with the norm:

$$\|u\|_m = \begin{cases} \|\Delta^k u\|_{L^2(\Omega)} & \text{if } m = 2k, \\ \|\nabla(\Delta^k u)\|_{L^2(\Omega)} & \text{if } m = 2k+1. \end{cases}$$

Throughout this paper, we assume that $N \geq 2m + 1$. The first eigenvalue of $(-\Delta)^m$ in Ω with Navier boundary conditions and the one of $(-\Delta)^m$ in B_R with Dirichlet boundary conditions will be denoted indifferently by Λ_1 and we denote by λ_1 the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

The main goal of this paper is to improve the results of [16] under some weaker conditions. One of the key ingredients in the case of Eq. (1.1) is the L^∞ boundary estimates of gradient derived in [16] (see pages 469–471). For this purpose, Soranzo adapted the proof of the idea in [5], by using Troy's techniques [17]. More precisely, since Eq. (1.1) may be written as the following system:

$$\begin{cases} u_0 = u, \\ -\Delta u_0 = u_1, \\ \vdots \\ -\Delta u_{m-1} = f(u_0), \\ u_0 = \cdots = u_{m-1} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

adapting the moving plan technique used by Troy in his famous paper [17], we can prove that u_i are decreasing with respect to the outward normal direction in a neighbourhood of the boundary. Clearly, this allows us to get estimate of L^∞ norm of ∇u_i near the boundary by interior estimates on u_i. Comparing to the Laplacian operator [5], the system case is more restrictive and one has to assume that $f \in C^1(\mathbb{R}_+)$ with $f' \geq 0$. However, an examination of the proof of Lemma 1.1 in [16] shows that we may relax to the following assumption: