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Random dynamical systems generated by stochastic Navier–Stokes equations on a rotating sphere

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In this paper we first prove the existence and uniqueness of the solution to the stochastic Navier–Stokes equations on the rotating 2-dimensional unit sphere. Then we show the existence of an asymptotically compact random dynamical system associated with the equations.

1. Introduction

The aim of this work is to initiate systematic analysis of rotating stochastic fluids on surfaces, notably on a sphere, and in thin shells. Importance of such problems for geophysical fluid dynamics and climate modelling is well known. This paper is concerned with the following stochastic Navier–Stokes equations (SNSEs) on a 2-dimensional rotating sphere:

$$\partial_t \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} - \nu \mathbf{L}\mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} + \nabla p = \mathbf{f} + n(\mathbf{x}, t), \qquad \operatorname{div} \mathbf{u} = 0, \tag{1.1}$$

where $\mathbf{L}$ is the stress tensor, $\boldsymbol{\omega}$ is the Coriolis acceleration, $\mathbf{f}$ stands for the external force and n is the Gaussian noise, white in time and correlated in space. Rigorous definitions of all the quantities in this equation are given in Sections 2 and 3. We will answer two fundamental questions about Eq. (1.1). The first one is about the existence and uniqueness of appropriately defined solutions and the second one is about the existence of stochastic flow and its asymptotic compactness.

The deterministic Navier–Stokes equations (NSEs) on a sphere have been an object of intense study since early 1990s. Il'in and Filatov [34,32] considered the existence and uniqueness of solutions to these equations

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and estimated the Hausdorff dimension of their global attractors [33]. Temam and Wang [49] considered the inertial forms of NSEs on spheres while Temam and Ziane [50], see also [3], proved that the NSEs on the 2-dimensional unit sphere are a limit of NSEs defined on spherical shells [50]. In other directions, Cao, Rammaha and Titi [14] proved the Gevrey regularity of the solution and found an upper bound on the asymptotic degrees of freedom for the long-time dynamics.

Concerning the numerical simulation of the deterministic NSEs on a sphere, Fengler and Freeden [21] obtained some impressive numerical results using the spectral method, while the numerical analysis of a pseudo-spectral method for these equations has been carried out by Ganesh, Le Gia and Sloan in [26].

The question of the existence and uniqueness of solutions to the stochastic Navier–Stokes equations (SNSEs) on 2D bounded domains has been thoroughly investigated by many authors, beginning with a 1973 paper [4] by Bensoussan and Temam. In the case when the continuous dependence on initial data remains open (for example, when the initial data is merely in L^2), the existence of martingale solutions has been considered by Capiński and Gatarek [15], Flandoli and Gatarek [23,24] and Mikulevicius and Rozovskii [38]. The uniqueness of the martingale solution for the SNSEs on flat 2D bounded domains has been proved by Ferrario in [22]. When working in spaces where continuous dependence on the initial data can be expected, the existence of solutions can sometimes be established on a preordained probability space. Such solutions are often referred to as "strong" (in probabilistic sense) or "pathwise" solutions. The existence of pathwise solutions in $L^\infty([0,T],L^2)$ has been established by Da Prato and Zabczyk [18] and later by others [5,36]. The existence of local solutions evolving in Sobolev spaces, such as $W^{1,p}$, was addressed by Mikulevicius and Rozovskii in [37] and Brzeźniak and Peszat in [11].

The existence and uniqueness of pathwise variational solutions to the SNSE on the rotating sphere for a general Gaussian noise is proved in Theorem 3.2. Let us describe briefly the main ideas of this proof since they are also important for the proof of our second main result, Theorem 6.11. Using the Leray–Helmholtz projection we reformulate first equation (1.1) as an abstract Itô equation in $H = L^2(\mathbb{S}^2)$:

$$d\mathbf{u}(t) + \mathbf{A}\mathbf{u}(t)dt + \mathbf{B}\big(\mathbf{u}(t),\mathbf{u}(t)\big)dt + \mathbf{C}\mathbf{u} = \mathbf{f}dt + GdW(t), \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{1.2}$$

Then, for any $\alpha > 0$, we introduce a stationary Ornstein–Uhlenbeck process $\mathbf{z}_\alpha$ defined as a solution to the linear equation

$$d\mathbf{z}_\alpha + (\nu\mathbf{A} + \mathbf{C} + \alpha)\mathbf{z}_\alpha dt = GdW(t), \quad t \in \mathbb{R}, \tag{1.3}$$

where $W(t)$, $t \in \mathbb{R}$, is a two-sided analog of the Wiener process $W(t)$, $t \geq 0$, from (1.2). Let us note that $\alpha > 0$ is arbitrary for the proof of the existence and uniqueness but it will have to be judiciously chosen for our next main result. We will show that the process $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{z}_\alpha(t)$ is a solution to the problem:

$$\begin{cases} \partial_t\mathbf{v} = -\nu\mathbf{A}\mathbf{v} - \mathbf{B}(\mathbf{v}+\mathbf{z}_\alpha,\mathbf{v}+\mathbf{z}_\alpha) - \mathbf{C}\mathbf{v} + \alpha\mathbf{z}_\alpha + \mathbf{f}, \\ \mathbf{v}(0) = \mathbf{v}_0. \end{cases} \tag{1.4}$$

The last equation is deterministic for each fixed trajectory of the process $\mathbf{z}_\alpha$ and the facts that it is considered on a sphere and the presence of the Coriolis force lead to minor modifications in the proof of existence. We can still use the classical Galerkin arguments based on expansions into series of vector spherical harmonics. The only new condition we require is, roughly speaking, that the solution $\mathbf{z}_\alpha$ to Eq. (1.3) has an L^4-solution. Let us note that transformation of the stochastic equation (1.2) for $\mathbf{u}$ to a deterministic equation (1.4) for $\mathbf{v}$ and stationarity of $\mathbf{z}_\alpha$ are also crucial for the proof of our second main result in Theorem 6.11. In order to prove uniqueness we modify the classical argument of Lions and Prodi [35]. Next, in Theorem 3.3 we prove continuous dependence on the force and the driving noise.

The main result of this paper is presented in Theorem 6.11, where we prove that the random dynamical system associated to Eq. (3.4) is asymptotically compact. This result is crucial for the proof of the existence