J. Math. Anal. Appl. 426 (2015) 585–593

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the density of classes of closed convex sets with pointwise constraints in Sobolev spaces ☆

M. Hintermüller, C.N. Rautenberg *

Institute for Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

ARTICLE INFO

Article history:
Received 26 September 2013
Available online 28 January 2015
Submitted by A. Dontchev

Keywords:
Closed convex sets
Dense embedding
Gradients
Pointwise constraints
Sobolev spaces

ABSTRACT

For a Banach space X of $\mathbb{R}^M$-valued functions on a Lipschitz domain, let $\mathbf{K}(X) \subset X$ be a closed convex set arising from pointwise constraints on the value of the function, its gradient or its divergence, respectively. The main result of the paper establishes, under certain conditions, the density of $\mathbf{K}(X_0)$ in $\mathbf{K}(X_1)$ where X_0 is densely and continuously embedded in X_1. The proof is constructive, utilizes the theory of mollifiers and can be applied to Sobolev spaces such as $H_0(\mathrm{div}, \Omega)$ and $W_0^{1,p}(\Omega)$, in particular. It is also shown that such a density result cannot be expected in general.

1. Introduction

Many problems in the calculus of variations involve, either directly or through (Fenchel) dualization, constraint sets of the type

$$\mathbf{K}(X) := \left\{ \mathbf{f} \in X : \left|(G\mathbf{f})(x)\right| \leq \alpha(x) \text{ a.e., } x \in \Omega \right\},$$

where $\Omega \subset \mathbb{R}^N$ represents some underlying domain, $N \in \mathbb{N}$, X is a Banach space of functions on Ω, $|\cdot|$ stands for the Euclidean norm, α denotes a sufficiently regular function with $\alpha(x) \geq \underline{\alpha} > 0$ for $x \in \Omega$, and where "a.e." stands for "almost everywhere". Further, the operator G takes one of the following choices:

$$G = \mathrm{id}, \qquad G = \nabla, \qquad G = \mathrm{div}\,.$$

☆ This research was supported by the Austrian Science Fund FWF through START-Projekt Y305 "Interfaces and Free Boundaries", the FWF-SFB F32 04-N18 "Mathematical Optimization and Its Applications in Biomedical Sciences", the DFG Research Center MATHEON through Project C28, and the MATHEON Projects SE5, SE15 and OT1 funded by the Einstein Center for Mathematics Berlin (ECMath). This research was partially conducted when Carlos N. Rautenberg was part of the Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8010 Graz, Austria.
* Corresponding author.
E-mail addresses: hint@math.hu-berlin.de (M. Hintermüller), carlos.rautenberg@math.hu-berlin.de (C.N. Rautenberg).

http://dx.doi.org/10.1016/j.jmaa.2015.01.060
0022-247X/© 2015 Elsevier Inc. All rights reserved.

Let X_1 denote, for instance, a Hilbert space of $\mathbb{R}^M$-valued functions over Ω and X_0 a Banach space which is continuously and densely embedded in X_1. For approximation purposes it is often necessary to find an answer to the following question:

$$\text{Is } \mathbf{K}(X_0) \text{ is dense in } \mathbf{K}(X_1) \text{ with respect to the norm in } X_1\text{?} \tag{1}$$

In general, the answer is not positive. In fact, $\mathbf{K}(X_0)$ might not even contain non-trivial elements as the following example[1] demonstrates.

A counter-example. Consider $X_1 = L^2(\Omega)^M$, where Ω is a domain in $\mathbb{R}^N$. Let $\{p_n\}_{n=1}^{\infty}$ be an enumeration of a dense set in Ω and $\phi_n(x) := |x - p_n|^{-1/4}$ for $x \neq p_n$. Note that since Ω is bounded, there exists $K > 0$ such that $|\phi_n|_{L^2(\Omega)} \leq K$ for all $n \in \mathbb{N}$. Then, $g := \sum_{k=1}^{\infty} k^{-2}\phi_k$ belongs to $L^2(\Omega)$, is strictly positive on $\overline{\Omega}$, and it is unbounded at each p_n, i.e., it is unbounded on a dense set.

Let X_0 be the linear space of functions $\mathbf{f} = \mathbf{h}g$ with $\mathbf{h} \in C(\overline{\Omega})^M$ and endowed with norm $|\mathbf{f}|_{X_0} = \sup_{x\in\Omega} |\mathbf{h}(x)|$, X_0 is a Banach space. Clearly, if $\mathbf{f} \in X_0$, then $\mathbf{f} \in X_1$ and $|\mathbf{f}|_{X_1} \leq |g|_{X_1}|\mathbf{h}|_{C(\overline{\Omega})^M} = |g|_{X_1}|\mathbf{f}|_{X_0}$, proving the embedding to be continuous. Let $\mathbf{f} \in C(\overline{\Omega})^M$ be arbitrary, and let $\mathbf{f}_n \in X_1$ be of the form $\mathbf{f}_n = \mathbf{h}_n g$ with $\mathbf{h}_n(x) = \mathbf{f}/g_n$ and $g_n = \sum_{k=1}^{\infty} k^{-2}\min(\phi_k, n) \in C(\overline{\Omega})$, where the min-operation is understood in a pointwise sense. Then it holds that

$$|\mathbf{f}_n - \mathbf{f}|_{X_1} = |\mathbf{h}_n g - \mathbf{f}|_{X_1} = |\mathbf{h}_n g - \mathbf{h}_n g_n|_{X_1} \leq |\mathbf{h}_n|_{C(\overline{\Omega})^M}|g - g_n|_{L^2(\Omega)}.$$

One readily observes that $|g - g_n|_{L^2(\Omega)} \to 0$ since we have $|\min(\phi_k, n) - \phi_k|_{L^2(\Omega)} \to 0$ for each k. Therefore, X_0 is dense in $C(\overline{\Omega})^M$ with respect to the X_1-norm, and since the latter is dense in X_1, X_0 is also dense in X_1. Finally, for $G = \mathrm{id}$ and an arbitrary $\alpha \in C(\overline{\Omega})$, we have that $\mathbf{K}(X_0) = \{0\}$ which is clearly not dense in $\mathbf{K}(X_1)$. Therefore, the dense and continuous embedding of X_0 in X_1 is not sufficient for

$$\overline{\mathbf{K}(X_0)}^{X_1} = \mathbf{K}(X_1). \tag{2}$$

Motivating applications. We briefly mention several motivating applications where (1) emerges from Fenchel dualization [7] and semismooth Newton solvers [11]. In a rather abstract setting, *regularized* total variation type image restoration [14], energies related to Bingham fluids [10], simplified friction problems or elasto-plastic problems in material science [6,16,4,18,13] can be associated with the following problem:

$$\text{minimize} \quad F(\mathbf{y}) + \alpha \int_S \left|C\mathbf{y}(s)\right| \mathrm{d}s \quad \text{over } \mathbf{y} \in Y, \tag{3}$$

where Y denotes a real Banach space with topological dual Y^*, $F : Y \to \mathbb{R}$ is of the form $F(\mathbf{y}) = \frac{1}{2}\langle A(\mathbf{y} - f), \mathbf{y} - f\rangle_{Y^*,Y} + \langle a, \mathbf{y}\rangle_{Y^*,Y} + b$ where $\langle\cdot,\cdot\rangle_{Y^*,Y}$ denotes the duality pairing between Y and Y^*, $A \in \mathcal{L}(Y, Y^*)$ is invertible, $a \in Y^*$ and $b \in \mathbb{R}$. Furthermore, $\alpha > 0$ is fixed and C is a linear and continuous operator from Y to $L^2(S, \mathbb{R}^{L\times M})$, i.e., $C \in \mathcal{L}(Y, L^2(S, \mathbb{R}^{L\times M}))$, with $L \in \mathbb{N}$, $1 \leq L, M \leq N$, and $S \subseteq \Omega$ or $S \subseteq \partial\Omega$, where $\partial\Omega$ denotes the boundary of $\Omega \subset \mathbb{R}^N$. We emphasize here that the functional associated with $\alpha > 0$ in (3) changes in the context of the (non-regularized) total variation based image restoration: see (6) and (7) below, where $Y = BV(\Omega)$. The Fenchel dual problem of (3) is given by

$$\text{minimize} \quad F^*\left(C^*\mathbf{p}\right) \quad \text{over } \mathbf{p} \in L^2\left(S, \mathbb{R}^{L\times M}\right) \tag{4a}$$

$$\text{subject to} \quad \left|\mathbf{p}(x)\right| \leq \alpha \quad \text{a.e., } x \in S. \tag{4b}$$

[1] The construction is based on an idea of Martin Hairer, Department of Mathematics, University of Warwick.