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Ramanujan–Hardy–Littlewood–Riesz phenomena for Hecke forms

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ABSTRACT

We generalize a result of Ramanujan, Hardy and Littlewood to the setting of primitive Hecke forms, which interestingly exhibits faster convergence than in the initial case of the Riemann zeta function. We also provide a criterion in the spirit of Riesz for the Riemann Hypothesis for the associated L-functions.

1. Introduction

Ramanujan's Notebooks [26,3,4] and his Lost Notebook [27,2] are filled with many striking results. An example of this is the following formula discussed in [4, p. 470] involving infinite series of the Möbius function:

For any real number $p > 0$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)e^{-p/n^2}}{n} = \sqrt{\frac{\pi}{p}} \sum_{n=1}^{\infty} \frac{\mu(n)e^{-\pi^2/(n^2 p)}}{n}. \tag{1.1}$$

However, this formula is incorrect as it stands. During his stay in Cambridge, Ramanujan told Hardy and Littlewood about this identity, and later in [12, p. 156, Section 2.5] they corrected it as follows:

Let α and β be two positive numbers such that $\alpha\beta = \pi$. Assume that the series $\sum_\rho (\Gamma(\frac{1-\rho}{2})/\zeta'(\rho))a^\rho$ converges, where ρ runs through the non-trivial zeros of $\zeta(s)$ and a denotes a positive real number, and that the non-trivial zeros of $\zeta(s)$ are simple. Then

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$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha^2/n^2}-\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\beta^2/n^2}=-\frac{1}{2\sqrt{\beta}}\sum_{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}\beta^{\rho}. \tag{1.2}$$

The corrected identity (1.2) is discussed in detail in Berndt [4, p. 470], Paris and Kaminski [23, p. 143] and Titchmarsh [33, p. 219, Section 9.8]. Bhaskaran [5] has drawn connections of this formula with the Fourier reciprocity and Wiener's Tauberian theory. Some related additional results have been recently obtained in [9]. In (1.2), one does not actually need to assume convergence of the series on the right-hand side, one may instead bracket the terms satisfying

$$\left|\operatorname{Im}\rho-\operatorname{Im}\rho'\right|<\exp\bigl(-c\operatorname{Im}\rho/\log(\operatorname{Im}\rho)\bigr)+\exp\bigl(-c\operatorname{Im}\rho'/\log(\operatorname{Im}\rho')\bigr),$$

where c is a positive constant (see [12, p. 158] and [33, p. 220]). The local spacing distribution of zeros exhibits strong repulsion between consecutive zeros. After the pioneering work of Montgomery [21] on the pair correlation of zeros of the Riemann zeta function, higher level correlations for general L-functions have been explained by Rudnick and Sarnak [30], and Katz and Sarnak [15,16].

In (1.2), one still assumes simplicity of the zeros. This is widely believed to be true. The first 1.5×10^9 non-trivial zeros of the Riemann zeta function are on the critical line and are simple (see van de Lune, te Riele and Winter [19]). Also, from the work of Bui, Conrey and Young [6], who improved on earlier results by Selberg [31], Levinson [18], Heath-Brown [13] and Conrey [7], we know that at least 40.58% non-trivial zeros of the Riemann zeta function lie on the critical line and are simple. Actually, the simplicity conjecture is not really needed in the Ramanujan–Hardy–Littlewood result, in the sense that by an appropriate modification of the right-hand side of (1.2), one can prove an unconditional result:

For any positive numbers α and β such that $\alpha\beta=\pi$,

$$\begin{aligned}&\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha^2/n^2}-\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\beta^2/n^2}\\&=-\frac{1}{\sqrt{\beta}}\sum_{\rho}\frac{1}{(m_\rho-1)!}\cdot\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho)^{m_\rho}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\beta^{2s}\Big|_{s=\rho/2},\end{aligned} \tag{1.3}$$

where the summation on the right-hand side is understood in the sense of bracketing [33, p. 220], and m_ρ is the multiplicity of the zero ρ.

In the present paper, we obtain an analogue of the Ramanujan–Hardy–Littlewood conjecture for normalized primitive Hecke forms. Introducing a new parameter z, we also obtain a one-variable generalization which involves Bessel functions. Lastly, we provide a general criterion in the spirit of Riesz for the Riemann Hypothesis for L-functions attached to primitive Hecke forms.

Let χ be a primitive character modulo q. Let $M_k(q,\chi)$ (respectively $S_k(q,\chi)$) denote the space of modular forms (respectively cusp forms) of weight k, level q and nebentypus χ. Let $f\in S_k(q,\chi)$ be a primitive Hecke form, normalized by $a_f(1)=1$, so that the Fourier coefficients are the same as the Hecke eigenvalues (see for example [14, p. 372–373]). Let $\bar{f}$ denote the dual of f having Fourier expansion $\bar{f}(z):=\sum_{n=1}^{\infty}\overline{a_f(n)}e^{2\pi inz}\in S_k(q,\overline{\chi})$. Consider the associated Hecke L-function,

$$L(f,s)=\prod_p\bigl(1-a_f(p)p^{-s}+\chi(p)p^{k-1-2s}\bigr)^{-1}. \tag{1.4}$$

It has an analytic continuation to an entire function. The completed L-function $\Lambda(f,s):=(\frac{\sqrt{q}}{2\pi})^s\Gamma(s)L(f,s)$ satisfies the functional equation [14, p. 375, Theorem 14.17] $\Lambda(f,s)=i^k\overline{\eta}\Lambda(\overline{f},k-s)$. Here $\eta:=G(\overline{\chi})a_f(q)q^{-k/2}$, where $G(\chi)$ is the Gauss sum $G(\chi):=\sum_{m=1}^{q}\chi(m)e^{2\pi im/q}$. We work with the normalized Dirichlet series