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Characterization of compact geodesic spaces ☆

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A B S T R A C T

Driven by Klee's well-known work about the topological properties of convex sets in locally convex linear spaces, we give a complete characterization of compact geodesic spaces with curvature bounded below in terms of the fixed point property for continuous functions. Furthermore, we provide an example which highlights the role of the sectional curvature in our result.

1. Introduction

V.L. Klee gave in [7] a characterization of compactness for convex subsets of a locally convex vector space in terms of the fixed point property for continuous functions. He proved that if K is a convex subset of a locally convex metrizable topological linear space, the following assertions are equivalent: (1) K is compact; (2) K has the fixed point property; (3) no relatively closed subset of K is a topological ray. Our purpose in this work is to prove a counterpart of Klee's theorem for geodesic spaces with curvature bounded below.

Geodesic spaces with curvature bounded below can be understood as a generalization of Riemannian manifolds with sectional curvature bounded below. In recent years these spaces have attracted the attention of many researchers as they serve as very suitable frameworks to solve problems in different areas, apart from geometry, such as partial differential equation or abstract optimization. A very thorough discussion on these spaces can be found in [3,4], and a state of the art about their geometry can be found in the still work in process [1].

The fixed point property for continuous mapping in geodesic spaces has been studied by different authors. For instance, in [9] a direct proof of Brouwer's theorem was given in Hadamard manifolds and, as a consequence, the authors obtained a result on the existence of solutions for a variational inequality. In [2] the

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authors obtained a direct proof of Schauder's theorem in a class of geodesic metric spaces which contains, among others, Banach spaces, Busemann spaces and CAT(κ) spaces with $\kappa > 0$ and diameter strictly less than $D_\kappa/2$. On the other hand, W.A. Kirk [6] proved that every geodesically bounded $\mathbb{R}$-tree has the fixed point property for continuous mappings. Since $\mathbb{R}$-trees are CAT(0) spaces without bounded below curvature, it is natural to ask whether we can obtain Klee's result in geodesic spaces if we impose some restriction about the curvature. In this paper we prove, see Theorem 10, that complete uniquely geodesic spaces with curvature bounded from below, and which satisfy a mild condition on continuity of midpoints, have the fixed point property if and only if they are compact. The main step in the proof of our result is that if X is a complete uniquely geodesic space with curvature bounded below, then X is compact if and only if X does not contain a closed polygonal ray.

In the last section of the paper, using gluing technique, we construct a noncompact CAT(0) space which does not contain a closed polygonal ray. The analysis of this example suggests the possibility of weakening the hypothesis on the lower curvature bound in our main theorem.

2. Preliminaries

We start by fixing notations and recalling some basic facts about geodesic metric spaces. Let (X, d) be a metric space and let $x, y \in X$. A *geodesic* joining x to y is a mapping $\gamma : [0, l] \subseteq \mathbb{R} \to X$ such that $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, l]$. The image $\gamma([0, l])$ of γ is called a *geodesic segment* joining x and y. A metric space (X, d) is said to be a *uniquely geodesic space* if every two distinct points x and y of X are joined by a unique geodesic. For the sake of simplicity we use $[x, y]$ to denote the unique geodesic segment joining x and y. A subset A of a uniquely geodesic space is said to be *convex* if given $x, y \in A$ the segment $[x, y]$ is contained in A.

The continuity of the distance between midpoints of geodesic segments has been shown to be an important property in the study of the fixed point property in geodesic spaces. A uniquely geodesic space (X, d) is said to have property (P), see [2], if

$$\lim_{\varepsilon \searrow 0} \sup\big\{d\big((1-t)x \oplus ty, (1-t)x \oplus tz\big)\colon\ t \in [0,1],\ x, y, z \in X,\ d(y,z) \le \varepsilon\big\} = 0, \tag{P}$$

where $(1 - t)x \oplus ty$ denotes the unique point in $[x, y]$ such that $d(x, (1 - t)x \oplus ty) = td(x, y)$.

The following version of Schauder's theorem in geodesic spaces with property (P) was proved in [2].

Theorem 1. *Let (X, d) be a uniquely geodesic space such that it satisfies property (P) and all balls are convex. Let K be a nonempty closed convex subset of (X, d). Then, any continuous mapping $T : K \to K$ with compact range $\overline{T(K)}$ has a fixed point in K.*

Example 2.

1. Strictly convex normed spaces are uniquely geodesic spaces with property (P).
2. Recall that a metric space (X, d) is a *hyperbolic space* (in the sense of Reich–Shafrir [13]) if X is uniquely geodesic and the following inequality holds:

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \le \frac{1}{2}d(y,z) \quad \text{for all } x, y, z \in X. \tag{1}$$

   Using induction and a continuity argument, it can be proved that (1) is equivalent to the following inequality:

$$d\big((1-t)x \oplus ty, (1-t)w \oplus tz\big) \le (1-t)d(x,w) + td(y,z), \tag{2}$$