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Bounds for α-optimal partitioning of a measurable space based on several efficient partitions

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ABSTRACT

We provide a two-sided inequality for the α-optimal partition value of a measurable space according to a finite number of nonatomic finite measures. The result extends and often improves Legut [Inequalities for α-optimal partitioning of a measurable space, Proc. Amer. Math. Soc. 104 (1988)] since the bounds are obtained considering several partitions that maximize the weighted sum of the partition values with varying weights, instead of a single one. Furthermore, we show conditions that make these bounds sharper.

1. Introduction

Let $(C, \mathcal{C})$ be a measurable space, $\mu_1, \dots, \mu_n$ be n nonatomic finite measures defined on the same σ-algebra $\mathcal{C}$, and let $\mathscr{P}$ be the set of all measurable partitions $(A_1, \dots, A_n)$ of C ($A_i \in \mathcal{C}$ for all $i = 1, \dots, n$, $\bigcup_{i=1}^n A_i = C$, $A_i \cap A_j = \emptyset$ for all $i \neq j$). Let Δ_{n-1} denote the $(n-1)$-dimensional simplex. For this definition, and the many others taken from convex analysis, we refer to [9].

Definition 1. A partition $(A_1^*, \dots, A_n^*) \in \mathscr{P}$ is said to be α-optimal, for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T \in \operatorname{ri} \Delta_{n-1}$, the relative interior of Δ_{n-1}, if

$$v^\alpha := \min_{i=1,\dots,n} \left\{ \frac{\mu_i(A_i^*)}{\alpha_i} \right\} = \sup \left\{ \min_{i=1,\dots,n} \left\{ \frac{\mu_i(A_i)}{\alpha_i} \right\} : (A_1, \dots, A_n) \in \mathscr{P} \right\}. \tag{1}$$

This problem has a consolidated interpretation in mathematical economics. We adopt the model considered in Dubins and Spanier [6]. C is a non-homogeneous, infinitely divisible good to be distributed

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among n agents with idiosyncratic preferences, represented by the measures. A partition $(A_1, \dots, A_n) \in \mathscr{P}$ describes a possible division of the good, with portion A_i (not necessarily connected) given to agent i. A satisfactory compromise between the conflicting interests of the agents, each having a relative claim α_i, $i = 1, \dots, n$, over the cake, is given by the α-optimal partition. It can be shown that the proposed solution coincides with the Kalai–Smorodinsky solution for bargaining problems (see Kalai and Smorodinsky [11] and Kalai [10]). When $\{\mu_i\}_{i=1,\dots,n}$ are all probability measures, i.e. $\mu_i(C) = 1$ for all $i = 1, \dots, n$, the claim vector $\boldsymbol{\alpha} = (1/n, \dots, 1/n)^T$ describes a situation of perfect parity among agents. The necessity to consider finite measures stems from game theoretic extensions of the models. For such extensions we refer to Legut [13], Legut et al. [14] and Dall'Aglio et al. [4].

When all the μ_i are probability measures, Dubins and Spanier [6] showed that if $\mu_i \neq \mu_j$ for some $i \neq j$, then $v^\alpha > 1$. This bound was improved, together with the definition of an upper bound by Elton et al. [8]. A further improvement for the lower bound was given by Legut [12]. More recently, Legut and Wilczyńsky [16] gave an explicit formula for the value of v^α (and of the corresponding optimal partition) for the case $n = 2$, based on the Neyman–Pearson lemma.

The aim of the present work is twofold: We provide further refinements for Legut's bounds for any n, and we show conditions that make these bounds sharper. We consider here the same geometrical setting employed by Legut [12], i.e. the partition range, also known as Individual Pieces Set (IPS) (see Barbanel [2] for a thorough review of its properties), defined as

$$\mathcal{R} := \{(\mu_1(A_1), \dots, \mu_n(A_n)) : (A_1, \dots, A_n) \in \mathscr{P}\} \subset \mathbb{R}^n_+.$$

Let us consider some of its features. The set $\mathcal{R}$ is compact and convex (see Dvoretzky et al. [7]). The supremum in (1) is therefore attained. Moreover, as shown by Legut and Wilczyńsky [15],

$$v^\alpha = \max\{r \in \mathbb{R}_+ : (r\alpha_1, r\alpha_2, \dots, r\alpha_n)^T \cap \mathcal{R} \neq \emptyset\}. \tag{2}$$

So, the vector $(v^\alpha\alpha_1, \dots, v^\alpha\alpha_n)^T$ is the intersection between the Pareto frontier of $\mathcal{R}$ and the ray $r\boldsymbol{\alpha} = \{(r\alpha_1, \dots, r\alpha_n)^T : r \geq 0\}$.

To find both bounds, Legut locates the solution of the maxsum problem $\sup\{\sum_{i=1}^n \mu_i(A_i) : (A_1, \dots, A_n) \in \mathscr{P}\}$ on the partition range. Then, he finds the convex hull of this point with the corner points of the partition range to find a lower bound, and uses a separating hyperplane argument to find the upper bound. We keep the same framework, but consider the solutions of several maxsum problems with weighted coordinates to find better approximations. Fix $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)^T \in \Delta_{n-1}$ and consider

$$\sum_{i=1}^n \beta_i \mu_i(A_i^\beta) = \sup\left\{\sum_{i=1}^n \beta_i \mu_i(A_i) : (A_1, \dots, A_n) \in \mathscr{P}\right\}. \tag{3}$$

Let η be a non-negative finite-valued measure with respect to which each μ_i is absolutely continuous (for instance we may consider $\eta = \sum_{i=1}^n \mu_i$). Then, by the Radon–Nikodym theorem, for each $A \in \mathcal{C}$,

$$\mu_i(A) = \int_A f_i \, d\eta \quad \forall i = 1, \dots, n,$$

where f_i is the Radon–Nikodym derivative of μ_i with respect to η.

Finding a solution for (3) is straightforward:

Proposition 1. *(See [6, Theorem 2], [1, Theorem 2], [3, Proposition 4.3].) Let $\boldsymbol{\beta} \in \Delta_{n-1}$ and let $B^\beta = (A_1^\beta, \dots, A_n^\beta)$ be a partition of C. If*