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The role of generalized convexity in conic copula constructions

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ABSTRACT

Inspired by the notion of a conic semi-copula, we introduce upper conic, lower conic and biconic semi-copulas with a given section. Such semi-copulas are constructed by linear interpolation on segments connecting the graph of a strict negation operator to the points $(0, 0)$ and/or $(1, 1)$. The important subclasses of upper conic, lower conic and biconic (quasi-)copulas with a given section are characterized. The notion of generalized convexity turns out to play a key role in this characterization. Some examples are also provided.

1. Introduction

The notion of a copula appeared for the first time in the field of probability theory. Copulas turn out to be appropriate tools for linking a joint distribution function with its margins [19,34]. The classes of semi-copulas and quasi-copulas extend the class of copulas and they are important in many application domains. For instance, in the field of reliability theory, semi-copulas turn out to be appropriate tools for capturing the relation between multivariate aging and dependence [2,11]. Another prominent application of semi-copulas is in fuzzy set theory, where they gain their importance from the fact that they extend the Boolean conjunction [3].

Recall that a semi-copula [14,16,17] is a function $S : [0, 1]^2 \to [0, 1]$ satisfying the following conditions:

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(i) *boundary conditions*: for any $x \in [0,1]$, it holds that

$$S(x,0) = S(0,x) = 0, \qquad S(x,1) = S(1,x) = x;$$

(ii) *increasingness*: for any $x, x', y, y' \in [0,1]$ such that $x \le x'$ and $y \le y'$, it holds that $S(x,y) \le S(x',y')$.

The functions M and $T_{\mathbf{D}}$ given by $\mathrm{M}(x,y) = \min(x,y)$, and $T_{\mathbf{D}}(x,y) = \min(x,y)$ whenever $\max(x,y) = 1$, and $T_{\mathbf{D}}(x,y) = 0$ elsewhere, are examples of semi-copulas. Moreover, for any semi-copula S the inequality $T_{\mathbf{D}} \le S \le \mathrm{M}$ holds.

A semi-copula Q is a quasi-copula [4,18,28] if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0,1]$, it holds that

$$\left|Q\left(x', y'\right) - Q(x,y)\right| \le \left|x' - x\right| + \left|y' - y\right|.$$

A semi-copula C is a copula [1,26,30] if it is 2-increasing, i.e. for any $x, x', y, y' \in [0,1]$ such that $x \le x'$ and $y \le y'$, it holds that

$$V_C\left(\left[x, x'\right] \times \left[y, y'\right]\right) := C\left(x', y'\right) + C(x,y) - C\left(x', y\right) - C\left(x, y'\right) \ge 0.$$

$V_C([x,x'] \times [y,y'])$ is called the C-volume of the rectangle $[x,x'] \times [y,y']$. The (quasi-)copulas M and W, with $\mathrm{W}(x,y) = \max(x+y-1,0)$, are respectively the greatest and the smallest (quasi-)copula, i.e. for any (quasi-)copula C, the inequality $\mathrm{W} \le C \le \mathrm{M}$ holds. Another important copula is the product copula Π defined by $\Pi(x,y) = xy$.

Several methods to construct (semi-, quasi-)copulas have been introduced in the literature. Some of these methods start from given sections. Such sections can be the diagonal section and/or the opposite diagonal section [7,9,13,20–23,25], or a horizontal section and/or a vertical section [12,27,33]. All of the above methods use sections that are determined by straight lines in the unit square, such as the diagonal, the opposite diagonal, a horizontal line or a vertical line. In the present paper, we consider sections that are determined by a curve in the unit square, which represents a strict negation operator.

For any strict negation operator $N : [0,1] \to [0,1]$, the surface of M is constituted from (linear) segments connecting the points $(0,0,0)$ and $(a, N(a), \min(a, N(a)))$ as well as segments connecting the points $(a, N(a), \min(a, N(a)))$ and $(1,1,1)$, with $N(a) \le a$, and segments connecting the points $(0,0,0)$ and $(a, N(a), \min(a, N(a)))$ as well as segments connecting the points $(a, N(a), \min(a, N(a)))$ and $(1,1,1)$, with $N(a) \ge a$. This observation has motivated the construction presented in this paper.

This paper is organized as follows. In the following section, we recall some definitions and facts concerning convexity and generalized convexity. In Section 3, we introduce the class of upper conic functions with a given section. In Sections 4 and 5, we characterize upper conic semi-copulas, upper conic quasi-copulas and upper conic copulas with a given section. In Section 6 (resp. Section 7), we introduce in a similar way the classes of lower conic (resp. biconic) functions with a given section and characterize lower conic (resp. biconic) semi-copulas, lower conic (resp. biconic) quasi-copulas and lower conic (resp. biconic) copulas with a given section. Finally, we draw some conclusions.

2. Convexity and generalized convexity

Convexity plays a key role in the characterization of some classes of semilinear copulas, such as conic copulas [24] and biconic copulas [21]. A more general type of convexity, called *generalized convexity*, has been introduced in the literature and has been used, for instance, to characterize the comparability of two quasi-arithmetic means [5,32]. We denote an open, half-open or closed interval in $\mathbb{R}$ with lower endpoint a and upper endpoint b as $I(a,b)$.