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Remarks on a Brezis–Nirenberg's result ☆

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ABSTRACT

In this note, we revisit the equation with Sobolev critical exponent which is firstly studied by Brezis–Nirenberg in [4]:

$$(\mathcal{P})\quad \begin{cases} -\Delta u + \lambda u = \mu |u|^{p-2}u + \beta |u|^{2^*-2}u, & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 3$, $2 < p < 2^*$, $\beta > 0$, $\lambda, \mu \in \mathbb{R}$. We show that for any $m \in \mathbb{N}$, there exists $\beta_m > 0$ such that $(\mathcal{P})$ has m solutions for any $\lambda \in \mathbb{R}$, $\mu > 0$ and every $\beta \in (0, \beta_m)$. When $N \geq 4$, $\lambda < 0$, $\beta > 0$, we show that there is $\mu_0 > 0$ such that problem $(\mathcal{P})$ has a nontrivial solution for every $\mu > -\mu_0$. Moreover, we get multiple sign changing solutions for problem $(\mathcal{P})$.

1. Introduction

In the pioneering paper of Brezis–Nirenberg [4], they study the following equation:

$$\begin{cases} -\Delta u + \lambda u = \mu u^{p-1} + u^{2^*-1}, & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $2^* := \frac{2N}{N-2}$ $(N \geq 3)$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2 < p < 2^*$, $\lambda, \mu \in \mathbb{R}$ are parameters. They showed the following result.

Theorem A. *(See Corollaries 2.1–2.4 in [4].)*

(i) *Assume $N \geq 4$, $\mu > 0$, $\lambda \in (-\lambda_1(\Omega), 0)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, then problem (1.1) has a solution;*

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(ii) *Assume* $N = 3$, $\mu > 0$, $\lambda \in (-\lambda_1(\Omega), 0)$*:*
- *If* $4 < p < 6$*, then problem (1.1) has a solution;*
- *If* $2 < p \leq 4$*, then there exists* $\bar{\mu}_0 > 0$ *such that problem (1.1) has a solution for each* $\mu \geq \bar{\mu}_0$*.*

In this paper, we revisit the problem (1.1). More generally, we consider the following elliptic equation with critical exponent:

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{p-2}u + \beta |u|^{2^*-2}u, & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

We will prove the following theorems:

Theorem 1.1. *Assume that* $N \geq 4$, $\lambda < 0$, $\mu > 0$, $\beta > 0$*, then system (1.2) has a nontrivial solution. Furthermore, when* $\lambda \leq -\lambda_1(\Omega)$*, problem (1.2) has a sign changing solution.*

Moreover, we also have the following multiplicity result:

Theorem 1.2. *Assume that* $N \geq 3$, $\lambda \in \mathbb{R}$, $\mu > 0$, $\beta > 0$*, then for any* $m \in \mathbb{N}$*, there exists* $\beta_m > 0$ *such that problem (1.2) has* m *nontrivial solutions for every* $\beta \in (0, \beta_m)$*. Furthermore, when* $\lambda \leq -\lambda_1(\Omega)$*, system (1.2) has* m *sign changing solutions.*

Remark 1.1. Let $\mu = 1$ in Theorem 1.2 and set $u = av$ where $a > 0$ such that $a^{2^*-2}\beta = 1$, then problem (1.2) is transfered to the form of (1.1), i.e.,

$$\begin{cases} -\Delta v + \lambda v = \mu' |v|^{p-2}v + |v|^{2^*-2}v & \text{on } \Omega, \\ v = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\mu' = (\frac{1}{\beta})^{\frac{p-2}{2^*-2}}$. By Theorem 1.2, there exists $\mu_m > 0$ such that problem (1.3) has m solutions for each $\mu' > \mu_m$. The result somehow corresponds to that of (ii) in Theorem A.

In contrast with Theorem A and Theorems 1.1–1.2 where $\mu > 0$ is required, we can also get a solution when $\mu < 0$.

Theorem 1.3. *Assume that* $N \geq 4$, $\lambda < 0$, $\mu < 0$, $\beta > 0$*, then there exists* $\mu_0 > 0$ *such that problem (1.2) has a nontrivial solution for every* $\mu \in (-\mu_0, 0)$*.*

Remark 1.2. Let $\beta = 1$ in the equation of (1.2). By a transformation, the problem (1.2) turns into

$$\begin{cases} -\Delta v + \lambda v = -|v|^{p-2}v + \beta' |v|^{2^*-2}v & \text{on } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Then Theorem 1.3 implies that when $N \geq 4$, $\lambda < 0$, there exists $\beta_0 > 0$ such that problem (1.4) has a nontrivial solution for each $\beta' > \beta_0$.

When $\mu = 0$, $\beta = 1$, the following problem was studied in [4]:

$$\begin{cases} -\Delta u + \lambda u = u|u|^{2^*-2}, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.5}$$