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Blow-up criterion for the compressible magnetohydrodynamic equations with vacuum

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ABSTRACT

In this paper, the 3-D compressible magnetohydrodynamic (MHD) equations with initial vacuum or infinite electric conductivity is considered. We prove that the L^∞ norms of the deformation tensor $D(u)$ and the absolute temperature θ control the possible blow-up (see [18,23]) of strong solutions, especially for the non-resistive MHD system when the magnetic diffusion vanishes. This conclusion means that if a solution of the compressible MHD equations is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of $D(u)$ or θ as the critical time approaches. The viscosity coefficients are only restricted by the physical conditions. Our criterion (see (1.17)) is similar to [17] for 3-D incompressible Euler equations and to [12] for 3-D compressible isentropic Navier–Stokes equations.

1. Introduction

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly with each other, so the hydrodynamic and electrodynamic effects are coupled. The applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics and plasma physics. In 3-D space, the compressible magnetohydrodynamic equations in a domain Ω of $\mathbb{R}^3$ can be written as

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$$\begin{cases} H_t - \operatorname{rot}(u \times H) = -\operatorname{rot}\left(\frac{1}{\sigma}\operatorname{rot} H\right), \\ \operatorname{div} H = 0, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}\mathbb{T} + \operatorname{rot} H \times H, \\ (\rho\theta)_t + \operatorname{div}(\rho\theta u) - \kappa\Delta\theta + P\operatorname{div} u = \operatorname{div}(u\mathbb{T}) - u\operatorname{div}\mathbb{T} + \frac{1}{\sigma}|\operatorname{rot} H|^2. \end{cases} \tag{1.1}$$

In this system, $x \in \Omega$ is the spatial coordinate; $t \geq 0$ is the time; $H = (H^1, H^2, H^3)$ is the magnetic field; $\operatorname{rot} H = \nabla \times H$ denotes the rotation of the magnetic field; $0 < \sigma \leq \infty$ is the electric conductivity coefficient; ρ is the mass density; $u = (u^1, u^2, u^3) \in \mathbb{R}^3$ is the velocity of fluids; $\kappa > 0$ is the thermal conductivity coefficient; P is the pressure satisfying

$$P = R\rho\theta, \tag{1.2}$$

where θ is the absolute temperature, R is a positive constant; $\mathbb{T}$ is the viscosity stress tensor:

$$\mathbb{T} = 2\mu D(u) + \lambda \operatorname{div} u\mathbb{I}_3, \quad D(u) = \frac{\nabla u + (\nabla u)^\top}{2}, \tag{1.3}$$

where $D(u)$ is the deformation tensor, $\mathbb{I}_3$ is the 3×3 unit matrix, μ is the shear viscosity coefficient, $\lambda + \frac{2}{3}\mu$ is the bulk viscosity coefficient, μ and λ are both real constants satisfying

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0, \tag{1.4}$$

which ensures the ellipticity of the Lamé operator. Although the electric field E doesn't appear in system (1.1), it is indeed induced according to a relation

$$E = \frac{1}{\sigma}\operatorname{rot} H - u \times H$$

by moving the conductive flow in the magnetic field.

The aim of this paper is to give a blow-up criterion of strong solutions to system (1.1) in a bounded, smooth domain $\Omega \in \mathbb{R}^3$ with the initial condition:

$$(H, \rho, u, \theta)|_{t=0} = \big(H_0(x), \rho_0(x), u_0(x), \theta_0(x)\big), \quad x \in \Omega, \tag{1.5}$$

and the Dirichlet, Neumann boundary conditions for (H, u, θ):

$$(H, u, \partial\theta/\partial n)|_{\partial\Omega} = (0, 0, 0), \quad \text{when } 0 < \sigma < +\infty; \tag{1.6}$$

$$(u, \partial\theta/\partial n)|_{\partial\Omega} = (0, 0), \quad \text{when } \sigma = +\infty, \tag{1.7}$$

where n is the unit outer normal vector to $\partial\Omega$. Actually, some similar result for $\Omega = \mathbb{R}^3$ can be also obtained via the similar argument used in this paper.

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$D^{k,r} = \big\{f \in L^1_{loc}(\Omega) : |f|_{D^{k,r}} = \big|\nabla^k f\big|_{L^r} < +\infty\big\}, \qquad D^k = D^{k,2},$$

$$\big\|(f,g)\big\|_X = \|f\|_X + \|g\|_X, \qquad \|f\|_{1,0} = \|f\|_{H^1_0(\Omega)}, \qquad \|f\|_s = \|f\|_{H^s(\Omega)},$$

$$\|f\|_{W^{k,r}} = \|f\|_{W^{k,r}(\Omega)}, \qquad |f|_p = \|f\|_{L^p(\Omega)}, \qquad |f|_{D^{k,r}} = \|f\|_{D^{k,r}(\Omega)}.$$

A detailed study of homogeneous Sobolev space may be found in [9].