J. Math. Anal. Appl. 425 (2015) 983–1003

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Admissible fundamental operators ☆

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ARTICLE INFO

Article history:
Received 30 June 2014
Available online 12 January 2015
Submitted by J.A. Ball

Keywords:
Spectral set
Symmetrized bidisc
Γ-contraction
Fundamental operator
Admissible pair
Tetrablock

ABSTRACT

Let F and G be two bounded operators on two Hilbert spaces. Let their numerical radii be no greater than one. This note investigates when there is a Γ-contraction (S,P) such that F is the fundamental operator of (S,P) and G is the fundamental operator of (S^*,P^*). Theorem 1 puts a necessary condition on F and G for them to be the fundamental operators of (S,P) and (S^*,P^*) respectively. Theorem 2 shows that this necessary condition is also sufficient provided we restrict our attention to a certain special case. The general case is investigated in Theorem 3. Some of the results obtained for Γ-contractions are then applied to tetrablock contractions to figure out when two pairs (F_1,F_2) and (G_1,G_2) acting on two Hilbert spaces can be fundamental operators of a tetrablock contraction (A,B,P) and its adjoint (A^*,B^*,P^*) respectively. This is the content of Theorem 3.

1. Introduction

The symmetrized bidisc is

$$\Gamma = \big\{(z_1+z_2, z_1z_2) : |z_1|, |z_2| \le 1\big\}.$$

Its distinguished boundary, i.e., the Shilov boundary with respect to the algebra of functions continuous on Γ and holomorphic in the interior of Γ is $b\Gamma = \{(z_1+z_2, z_1z_2) : |z_1| = 1 = |z_2|\}$. A pair of commuting bounded operators (S,P) on a Hilbert space $\mathcal{H}$ having the symmetrized bidisc as a spectral set is called a Γ-contraction. This means that the joint spectrum $\sigma(S,P) \subset \Gamma$ and

$$\big\|f(S,P)\big\| \le \sup\big\{\big|f(s,p)\big| : (s,p) \in \Gamma\big\}$$

☆ This research is supported by the Department of Science and Technology, India through the project numbered SR/S4/MS:766/12 and the University Grants Commission, India via DSA-SAP.

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http://dx.doi.org/10.1016/j.jmaa.2015.01.006
0022-247X/

for all $f \in \mathbb{C}[z_1, z_2]$. The study of Γ-contractions was introduced and carried out very successfully over several papers by Agler and Young, see [3] and the references therein. It follows that the operator P is a contraction and $\|S\| \le 2$. It can be seen directly from the definition that (S^*, P^*) is a Γ contraction too. Let $D_P = (I - P^*P)^{1/2}$ and $\mathcal{D}_P = \overline{Ran} D_P$. The fundamental operator is the unique bounded operator on $\mathcal{D}_P$ that satisfies the fundamental equation

$$S - S^*P = D_P F D_P.$$

It has numerical radius $w(F)$ no greater than one. The fundamental operator of a Γ-contraction was introduced in [8]. There it is shown that the fundamental equation has a unique solution. The discovery of the fundamental operator of a Γ-contraction put a spurt in the activities around it. In particular, we would like to mention Sarkar's work [11] which made a significant contribution to the understanding of Γ-contractions.

In this paper, $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ will denote the algebra of all bounded operators on $\mathcal{H}$. Since (S^*, P^*) is also a Γ-contraction, it has its own fundamental operator $G \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(G) \le 1$. Note how both F and G feature in the following explicit construction of a boundary normal dilation.

A boundary normal dilation of a Γ-contraction (S, P) is a pair of commuting normal operators (R, U) on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that (R, U) is a *dilation* of the given pair (S, P) and $\sigma(R, U)$, the joint spectrum is contained in the distinguished boundary $b\Gamma$. *Dilation* means that

$$P_{\mathcal{H}} R^m U^n \big|_{\mathcal{H}} = S^m P^n.$$

Such a pair (R, U) is also called a Γ-unitary. The following construction, done by two of the authors of the present paper in [9] and independently by Pal in [10], is one of the very few explicit constructions of dilations known, the only other ones being Schaeffer's construction of the minimal unitary dilation of a contraction in [13] and Ando's construction of a commuting unitary dilation of a pair of commuting bounded operators in [4].

Known Theorem. *Let (S, P) be a Γ-contraction. Let F and G be the fundamental operators of (S, P) and (S^*, P^*) respectively. Consider the space $\mathcal{K}$ defined as*

$$\mathcal{K} = \cdots \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots.$$

Let R and U be defined on $\mathcal{K}$ as follows.

$$R = \left[\begin{array}{cccc|c|cccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & F & F^* & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & F & F^* & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & F & F^* D_P & -F^* P^* & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & S & D_{P^*} G & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & 0 & G^* & G & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & G^* & G & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & G^* & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right], \tag{1.1}$$