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The combined viscous semi-classical limit for a quantum hydrodynamic system with barrier potential

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ABSTRACT

We investigate the viscous model of quantum hydrodynamics, which describes the charge transport in a certain semiconductor. Quantum mechanical effects lead to third order derivatives, turning the stationary system into an elliptic system of mixed order in the sense of Douglis–Nirenberg. In the case most relevant to applications, the semiconductor device features a piecewise constant barrier potential. In the case of thermodynamic equilibrium, we obtain asymptotic expansions of interfacial layers of the particle density in neighbourhoods of the jump points of this barrier potential, and we present rigorous proofs of uniform estimates of the remainder terms in these asymptotic expansions.

1. Introduction

The ongoing miniaturisation of electronical devices requires the investigation of mathematical models for the electron transport that include quantum mechanical terms. One of these models is the isentropic viscous quantum hydrodynamic model

$$\begin{cases} \partial_t n - \operatorname{div} J = \nu \Delta n, \\ \partial_t J - \operatorname{div}\left(\dfrac{J \otimes J}{n}\right) - \nabla p(n) + n\nabla(V + V_B) + \dfrac{\varepsilon^2}{6} n \nabla \dfrac{\Delta\sqrt{n}}{\sqrt{n}} = \nu \Delta J - \dfrac{J}{\tau}, \\ \lambda^2 \Delta V = n - \mathcal{C}, \end{cases} \tag{1}$$

formulated for the unknown functions (n, J, V), and the independent variables are $t \in \mathbb{R}$ as time, and $x \in \mathbb{R}^d$ as space. The unknown functions are the (positive) scalar electron density n, the vectorial electric current density J, and the scalar electric potential V. The item $p(n)$ is a generic pressure term, and a common choice

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is $p(n) = Tn + \mu n$, with a temperature T given by a relation $T(n) = T_0 n^{\gamma-1}$ for a positive constant T_0 and some $\gamma \geq 1$, and $\mu > 0$. Furthermore, the barrier potential $V_B = V_B(x)$ and the doping profile $\mathcal{C} = \mathcal{C}(x)$ of the semiconductor are given functions that describe certain material properties; these two functions are typically piecewise constant, and they are of crucial importance for the working principle of devices as the resonant tunnel diode. The purpose of this paper is to study analytically the behaviour of the solutions (n, J, V) near the jump points of the barrier potential V_B.

Additionally, we have certain positive physical constants, which have been scaled for ease of notation: The Planck constant ε, a relaxation time τ, the Debye length λ, and a viscosity constant ν.

A model (1) without the viscosity terms on the right hand side was proposed in [11] as a variant of the classical Euler–Poisson system, augmented by a term

$$\frac{\varepsilon^2}{6} n \nabla \frac{\Delta\sqrt{n}}{\sqrt{n}} =: \frac{\varepsilon^2}{6} n \nabla B(n)$$

that involves the Bohm potential $B(n)$ and describes quantum mechanical effects. The expectation is that this term is negligible in those regions where the electron flow can be described in terms of classical physics (i.e., in some regions far away from jump points of V_B).

There are various ways to derive (inviscid) quantum hydrodynamic models; we mention the traditional moment method applied to the collision Wigner equation [11], an approach via WKB wave functions from the Schrödinger Poisson system [14], and the entropy minimization approach [16]. Augmenting the Wigner equation with a Fokker Planck operator that describes the interaction of the electrons with the phonons of the crystal lattice, the dissipation terms $\nu\Delta n$ and $\nu\Delta J$ appear, see [3]. For an overview of this field, we refer to [1] and [15].

The quantum mechanical effects enter the system mainly via the Bohm term $B(n)$, which introduces third order spatial derivatives into the momentum balance equation, which complicates analytical studies of (1) considerably, compare [4–6,10] for results on the transient problem without barrier potential. Further analytical difficulties arise from the barrier potential V_B having jumps, and in that situation the second equation of (1) must be understood in the distributional sense. We are not aware of any analytical results concerning the transient system (1), however we mention numerical simulations in [8,11,13,17,18], and [19].

We focus our attention to a one-dimensional, stationary system,

$$\begin{cases} J' = -\nu n'' \quad \text{in } [0,1], \\ 2\varepsilon^2 n \left(\dfrac{\sqrt{n}''}{\sqrt{n}}\right)' - \nu J'' - \big(p(n)\big)' + \dfrac{J}{\tau} = \left(\dfrac{J^2}{n}\right)' - n(V + V_B)' \quad \text{in } [0,1], \\ \lambda^2 V'' = n - \mathcal{C} \quad \text{in } [0,1]. \end{cases} \tag{2}$$

For such a stationary system (without barrier potential), the existence of solutions was shown in [12], assuming small applied voltages $V(1)$–$V(0)$ and small currents J, which corresponds to a subsonic condition for the moving electrons. Although formulated for the isothermal case $p(n) = (T_0 + \mu)n$, the results of [12] seem to generalize to the case of general pressure terms $p(n)$. And we also mention [9], where it was shown (in the isothermal case) that solutions $(n, V, J) \in W^{2,2}(0,1) \times W^{2,2}(0,1) \times W^{1,2}(0,1)$ to (2) for given (possibly large) Dirichlet boundary values for V and periodic boundary values for n do exist.

The purpose of the present paper is to extend the solution theory of [9] towards an asymptotic expansion of the solution, for vanishing values of the quantum mechanical parameters ε and ν, focusing on the equilibrium case. See also [20] for further results. It turns out that we find a similar asymptotic expansion of the particle density n as in [2,7] for a stationary quantum drift diffusion model.

The solution theory in [9] is based on a reformulation of the system (2) by means of a viscosity-adjusted Fermi level