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Linear and strong convergence of algorithms involving averaged nonexpansive operators

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ABSTRACT

We introduce regularity notions for averaged nonexpansive operators. Combined with regularity notions of their fixed point sets, we obtain linear and strong convergence results for quasicyclic, cyclic, and random iterations. New convergence results on the Borwein–Tam method (BTM) and on the cyclically anchored Douglas–Rachford algorithm (CADRA) are also presented. Finally, we provide a numerical comparison of BTM, CADRA and the classical method of cyclic projections for solving convex feasibility problems.

1. Overview

Throughout this paper, X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. The convex feasibility problem asks to find a point in the intersection of convex sets. This is an important problem in mathematics and engineering; see, e.g., [6,7,12–14,20,21,29], and the references therein.

Oftentimes, the convex sets are given as fixed point sets of projections or (more generally) averaged nonexpansive operators. In this case, weak convergence to a solution is guaranteed but the question arises under which circumstances can we guarantee strong or even linear convergence. The situation is quite clear for projection algorithms; see, e.g., [6] and also [23].

The aim of this paper is to provide verifiable sufficient conditions for strong and linear convergence of algorithms based on iterating convex combinations of averaged nonexpansive operators.

Our results can be nontechnically summarized as follows: *If each operator is well behaved and the fixed point sets relate well to each other, then the algorithm converges strongly or linearly.*

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Specifically, we obtain the following main results on iterations of averaged nonexpansive mappings:

- If each operator is boundedly linearly regular and the family of corresponding fixed point sets is boundedly linearly regular, then quasicyclic averaged algorithms converge linearly (Theorem 6.1).
- If each operator is boundedly regular and the family of corresponding fixed point sets is boundedly regular, then cyclic algorithms converge strongly (Theorem 7.11).
- If each operator is boundedly regular and the family of corresponding fixed point sets is innately boundedly regular, then random sequential algorithms converge strongly (Theorem 7.14).

We also focus in particular on algorithms featuring the Douglas–Rachford splitting operator and obtain new convergence results on the Borwein–Tam method and the cyclically anchored Douglas–Rachford algorithm.

The remainder of the paper is organized as follows. In Sections 2 and 3, we discuss (boundedly) linearly regular and averaged nonexpansive operators. The bounded linear regularity of the Douglas–Rachford operator in the transversal case is obtained in Section 4. In Section 5, we recall the key notions of Fejér monotonicity and regularity of collections of sets. Our main convergence result on quasicyclic algorithms is presented in Section 6. In Section 7, we turn to strong convergence results for cyclic and random algorithms. Applications and numerical results are provided in Section 8. Notation in this paper is quite standard and follows mostly [7]. The closed ball of radius r centred at x is denoted by $\operatorname{ball}(x;r)$.

2. Operators that are (boundedly) linearly regular

Our linear convergence results depend crucially on the concepts of (bounded) linear regularity which we introduce now.

Definition 2.1 *((Bounded) linear regularity).* Let $T: X \to X$ be such that $\operatorname{Fix} T \neq \varnothing$. We say that:

(i) T is *linearly regular* with constant $\kappa \geq 0$ if

$$(\forall x \in X) \quad d_{\operatorname{Fix} T}(x) \leq \kappa \|x - Tx\|. \tag{1}$$

(ii) T is *boundedly linearly regular* if

$$(\forall \rho > 0)\ (\exists \kappa \geq 0)\ \big(\forall x \in \operatorname{ball}(0;\rho)\big) \quad d_{\operatorname{Fix} T}(x) \leq \kappa \|x - Tx\|; \tag{2}$$

note that in general κ depends on ρ, which we sometimes indicate by writing $\kappa = \kappa(\rho)$.

We clearly have the implication

$$\text{linearly regular} \quad \Rightarrow \quad \text{boundedly linearly regular}. \tag{3}$$

Example 2.2 *(Relaxed projectors).* Let C be a nonempty closed convex subset of X and let $\lambda \in]0,2]$. Then $T = (1-\lambda)\operatorname{Id} + \lambda P_C$ is linearly regular with constant λ^{-1}.

Proof. Indeed, $\operatorname{Fix} T = C$ and $(\forall x \in X)$ $d_C(x) = \|x - P_C x\| = \lambda^{-1}\|x - Tx\|$. $\square$

The following example shows that an operator may be boundedly linearly regular yet not linearly regular. This illustrates that the converse of the implication (3) fails.