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On the dynamics of a strongly singular parabolic equation

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ABSTRACT

In this paper we study the semiflow defined by a semilinear parabolic equation, in which both the diffusion and the reaction term present strong order of singularity at the origin. We justify the existence of a global branch of nontrivial equilibrium solutions for subcritical nonlinearities. The approach, based on the critical Caffarelli–Kohn–Nirenberg inequality, follows the arguments of [21,22].

1. Introduction

We describe the dynamics of the semilinear parabolic problem

$$\begin{aligned} &\varphi_t = \nabla \cdot \left(|x|^{-(N-2)} \nabla \varphi\right) + \lambda |x|^{-r} \varphi - |x|^{-k} |\varphi|^{p-2} \varphi, \quad x \in \Omega,\ t > 0, \\ &\varphi(x,0) = \varphi_0(x), \quad x \in \Omega, \\ &\varphi(x,t) = 0 \quad \text{on } \partial\Omega,\ t > 0, \end{aligned} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter and $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an open bounded domain with sufficiently smooth boundary $\partial\Omega$. Motivated by the coexistence of the critical exponent $2-N$ and the singular reaction term, we assume, throughout the whole paper, that $0 \in \Omega$. The criticality is meant in the sense of the Caffarelli–Kohn–Nirenberg inequalities (cf. [12]). Existence, uniqueness and asymptotics results, for problems presenting singularities both at the origin and at infinity, are heavily connected to the aforementioned

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inequalities (cf. [1,9,13,15–17]). Indicatively, for the subcritical case, we refer to the weighted quasilinear stationary equation

$$-\nabla \cdot \left(|x|^{-p\gamma}|\nabla\varphi|^{p-2}\nabla\varphi\right) = f(x,\varphi), \tag{1.2}$$

in [1], where the critical singularity is not treated, since $\gamma < (N-p)/p$ and $1 < p < N$. In the same paper, non-existence results are established for certain nonlinearities f that are singular at the origin (see also [2,3] for corresponding evolution problems). In our case, the exponents are posed in the constrained field

$$\mathcal{C} := \left\{(r,k,p) : 0 \le r \le \frac{N-2}{2}, \frac{rp}{2} \le k \le \frac{N-2}{4}p, 2 < p < \frac{2N-2}{N-2}\right\}. \tag{1.3}$$

In the sequel, we define the weighted Sobolev space $\tilde{\mathcal{H}}$ as the completion of the $C_0^\infty(\Omega)$-functions under the norm

$$\|\varphi\|^2_{\tilde{\mathcal{H}}} := \int_\Omega |x|^{-(N-2)}|\nabla\varphi|^2 dx. \tag{1.4}$$

This space appears to be the natural energy space for (1.1). We note, by [31, Corollary 4.1], that $\|\cdot\|^2_{\tilde{\mathcal{H}}}$ is equivalent to the norm

$$N^2(\varphi) := \int_\Omega |x|^{-(N-2)}\left(|\varphi|^2 + |\nabla\varphi|^2\right) dx, \tag{1.5}$$

for all $\varphi \in C_0^\infty(\Omega)$. More details about the well-definition, approximation results and properties of $\tilde{\mathcal{H}}$ can be found in [31].

Our starting point is the analysis of the set of the equilibrium solutions of (1.1). The equilibria satisfy the semilinear elliptic problem

$$\begin{aligned} -\nabla \cdot \left(|x|^{-(N-2)}\nabla u\right) &= \lambda|x|^{-r}u - |x|^{-k}|u|^{p-2}u, \quad x \in \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.6}$$

Operator $\mathcal{L} := -\nabla\cdot(|x|^{-(N-2)}\nabla)$ defines an unbounded self-adjoint operator in $L^2(|x|^{-r}dx;\Omega)$ with compact inverse. Thus, a global branch of nonnegative solutions of (1.6), bifurcating from the trivial solution at $(\lambda_1, 0)$, where λ_1 is the positive principal eigenvalue of the linear eigenvalue problem

$$\begin{aligned} -\nabla \cdot \left(|x|^{-(N-2)}\nabla u\right) &= \lambda|x|^{-r}u, \quad x \in \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

is established. The main result of Section 2 is stated in the following.

Theorem 1.1. *The principal eigenvalue of (1.7), considered in $\tilde{\mathcal{H}}$, is a bifurcating point of (1.6) (in the sense of Rabinowitz), and C_{λ_1} is a global branch of nonnegative $\tilde{\mathcal{H}}$-solutions of (1.6).*

Following the standard procedure of the semiflow theory [8,19,29], we define a gradient semiflow in $\tilde{\mathcal{H}}$ for (1.1). The basic result of Section 3 is exactly the following.

Theorem 1.2. *The gradient semiflow $\mathcal{S}(t) : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$, induced by the evolution (1.1), possesses a global attractor $\mathcal{A}$ in $\tilde{\mathcal{H}}$. Let $\mathcal{E}$ denote the bounded set of the equilibrium points of $\mathcal{S}(t)$. For each complete orbit*