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Quenching of solutions to a class of semilinear parabolic equations with boundary degeneracy

Mingjun Zhou, Chunpeng Wang, Yuanyuan Nie *

School of Mathematics, Jilin University, Changchun 130012, China

A R T I C L E   I N F O

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A B S T R A C T

This paper concerns the quenching phenomenon of solutions to a class of semilinear parabolic equations with boundary degeneracy. In the case that the degeneracy is not strong, it is shown that there exists a critical length, which is positive, such that the solution exists globally in time if the length of the spatial interval is less than it, while quenches in a finite time if the length of the spatial interval is greater than it. Whereas in the case that the degeneracy is strong enough, the solution must be quenching in a finite time no matter how long the spatial interval is. Furthermore, for each quenching solution, the set of quenching points is determined and it is proved that its derivative with respect to the time must blow up at the quenching time.

1. Introduction

In this paper, we investigate the quenching phenomena of the solution to the following initial–boundary value problem

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(x^{\lambda}\frac{\partial u}{\partial x}\right)=f(u),\quad (x,t)\in(0,a)\times(0,T),\tag{1.1}$$

$$\left(x^{\lambda}\frac{\partial u}{\partial x}\right)(0,t)=0,\qquad u(a,t)=0,\quad t\in(0,T),\tag{1.2}$$

$$u(x,0)=0,\quad x\in(0,a),\tag{1.3}$$

where $\lambda>0$, $a>0$ and $f\in C^2([0,c))$ with $c>0$ satisfies

$$f(0)>0,\qquad f'(0)>0,\qquad f''(s)\geq 0\quad\text{for } 0<s<c,\qquad \lim_{s\to c^-}f(s)=+\infty.$$

* Corresponding author.
E-mail addresses: zhoumj12@mails.jlu.edu.cn (M. Zhou), wangcp@jlu.edu.cn (C. Wang), nieyuanyuan@live.cn (Y. Nie).

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As a typical equation with boundary degeneracy, (1.1) is degenerate at $x = 0$, a portion of the lateral boundary. Equations with degeneracy similar to the one in (1.1) are used to describe some models, such as the Black–Scholes model coming from the option pricing problem ([3] and hundreds of related papers), the Budyko–Sellers climate model [27] and a simplified Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate [6]. In very recent years, the null controllability of the control system has been governed by parabolic equations with boundary degeneracy. Particularly, for the control system governed by

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(x^\lambda \frac{\partial u}{\partial x}\right) = h(x,t)\chi_{(\omega_1,\omega_2)}, \quad (x,t) \in (0,1) \times (0,T),$$

it was proved that $\lambda = 2$ is a threshold for the null controllability in the sense that the control system is null controllable if $0 < \lambda < 2$ [2,6,7,25], while not if $\lambda \geq 2$ [5], where h is the control function, $\chi_{(\omega_1,\omega_2)}$ is the characteristic function of (ω_1, ω_2) with $0 < \omega_1 < \omega_2 < 1$. Furthermore, [28] studied the asymptotic behavior of solutions to the problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(x^\lambda \frac{\partial u}{\partial x}\right) &= u^p, \quad x > 0,\ t > 0, \\ x^\lambda \frac{\partial u}{\partial x}(0,t) &= 0, \quad t > 0, \\ u(x,0) &= u_0(x), \quad x > 0, \end{aligned}$$

and showed that its critical Fujita exponent is

$$p_c = \begin{cases} 3 - \lambda, & \text{if } 0 < \lambda < 2, \\ +\infty, & \text{if } \lambda \geq 2, \end{cases}$$

which implies that $\lambda = 2$ is still a threshold for the blow-up theorems of Fujita type.

Owing to the properties of f, the solution u to the problem (1.1)–(1.3) may quench, i.e., there exists a time $0 < T_* \leq +\infty$ such that

$$\sup_{(0,a)} u(\cdot,t) < c \quad \text{for each } 0 < t < T_* \quad \text{and} \quad \lim_{t \to T_*^-} \sup_{(0,a)} u(\cdot,t) = c.$$

It is called that u quenches in a finite time if $T_* < +\infty$, while u quenches in infinite time if $T_* = +\infty$. Quenching phenomenon was first introduced by H. Kawarada [19] in 1975 for the problem (1.1)–(1.3) with $\lambda = 0$, where H. Kawarada proved the existence of the critical length. That is to say, the solution exists globally in time if the length of the spatial interval is less than it, while quenches in a finite time if the length of the spatial interval is greater than it. For the quenching case, H. Kawarada [19] also studied the set of quenching points and showed that the derivative of the solution with respect to the time blows up at the quenching time. However, it was unknown what happens when the length of the spatial interval is equal to the critical length in [19]. For the special case that

$$f(u) = \frac{1}{(c-u)^\beta}, \quad 0 \leq u < c \ (\beta > 0),$$

Levine [22] in 1989 proved that the solution cannot quench in infinite time by finding the explicit form of the minimum steady-state solution. It is referred to [13,14] for the general case. Since [19], there are many interesting results on quenching phenomena for semilinear uniformly parabolic equations (see, e.g., [1,4, 15,21,23,24]), singular or degenerate semilinear parabolic equations (see, e.g., [12,11,10,8,20]), quasilinear