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Stochastic initial boundary value problems subject to distributed and boundary noise and their optimal control

N.U. Ahmed

University of Ottawa, Canada

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A B S T R A C T

In this paper we consider a class of stochastic evolution equations arising from initial boundary value problems with both boundary and distributed noise. We prove existence and regularity of mild solutions. Then we consider a controlled version of the model and prove the existence of optimal controls and develop the necessary conditions of optimality for partially observed problems using relaxed controls.

1. Introduction

There are many papers written on stochastic maximum principle in finite dimensional spaces. The reader is referred to [7] and the extensive references given therein. However there are not many papers on the same topic for infinite dimensional spaces covering stochastic partial differential equations (SPDE). Some very recent and interesting results on this topic are covered in Refs. [4,5,12,15,16,19]. In [5] we consider semi-linear neutral stochastic evolution equations with controls in the drift and the diffusion operators and present necessary conditions of optimality. In [12] Duncan and Pasik-Duncan consider linear stochastic differential equations on Hilbert spaces with exponential-quadratic cost functionals giving differential operator Ricatti equations. They present also several interesting examples from initial boundary value problems. Fuhrman, Hu and Tessitore [15] present maximum principle for a class of stochastic partial differential equations subject to finite dimensional Brownian motion with controls appearing in the drift and the diffusion coefficients. The cost functional is of Bolza type. Fuhrman et al. consider regular controls and they develop second order necessary conditions. In [16] Hu and Peng develop some fundamental results on the question of existence and uniqueness of a large class of backward stochastic evolution equations (BSDE) on Hilbert spaces which we use in this paper. In [19] Zhou develops necessary conditions of optimality (maximum principle) for a very general class of linear non-degenerate (strictly elliptic) second order partial differential equations on a d-dimensional space with all the coefficients containing control. Except [12], the above mentioned papers do

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not consider boundary value problems which arise so naturally in all physical problems. For example, see [10] where Clason, Kaltenbacher, and Veljovic use boundary controls for Westervelt–Kuznetsov equation arising in acoustic and vibration problems. Maslowski [17] studied a general class of SPDES subject to both interior and boundary noise presenting a result on the question of existence and uniqueness of mild solutions. The main emphasis of the paper is stability dealing with the asymptotic properties of solutions such as exponential stability in the mean and existence and uniqueness of invariant measures. In [13], Duncan, Maslowski and Pasik Duncan consider the problem of adaptive boundary and point control of a class of linear stochastic systems on a Hilbert space. By using the fact that the stationary Riccati equation has a solution which is continuously dependent on the unknown parameter in the uniform operator topology they prove certainty equivalence adaptive control is selftuning.

Here in this paper we consider a general class of semilinear initial boundary value problems of even order with noise appearing both in the interior of the spatial domain as well as on its boundary. Using semigroup theory, we reformulate the stochastic partial differential equation (SPDE) as a stochastic evolution equation on a Hilbert space with the boundary noise term containing an unbounded operator. When the noise acts on the boundary the available results do not hold. This is one of the motivations of this paper. Further, we consider partially observed relaxed controls (measure valued stochastic processes) and prove existence of optimal controls and later develop the necessary conditions of optimality from which the maximum principle follows readily. The paper is organized as follows. In Section 2, we present the mathematical model of the system and reformulate this as an abstract stochastic evolution equation on Hilbert space. In Section 3, after basic assumptions are introduced, we prove the existence and regularity of mild solutions. Existence of optimal control is proved in Section 4. In Section 5 we present the necessary conditions of optimality. For illustration some examples are presented in Section 6. The paper is concluded with certain comments on open problems.

2. System model with distributed and boundary noise

A very large class of dynamic systems arising in physical sciences and engineering can be described by the following class of partial differential equations:

$$\begin{aligned} \partial\varphi/\partial t + \mathcal{A}\varphi &= f(t,\xi,\varphi) + \sigma(t,\xi,\varphi)V_d(t,\xi), \quad (t,\xi) \in I \times \Sigma, \\ (\mathcal{B}\varphi)(t,\xi) &= V_b(t,\xi), \quad (t,\xi) \in I \times \partial\Sigma \\ \varphi(0,\xi) &= \varphi_0(\xi), \quad \xi \in \Sigma \end{aligned} \tag{1}$$

subject to distributed and boundary noise $\{V_d, V_b\}$ defined on the domain $\Sigma \subset R^n$ and its boundary $\partial\Sigma$ respectively. The domain Σ is an open bounded set with smooth boundary and $I = (0,T]$ is an interval. The operator $\mathcal{A}$ is generally given by

$$(\mathcal{A}\varphi)(\xi) \equiv \sum_{|\alpha|\le 2m} a_\alpha(\xi) D^\alpha \varphi, \quad \text{on } \Sigma,$$

$$\text{with multi index } \alpha = \{\alpha_i\}_{i=1}^n,\ |\alpha| \equiv \sum_{i=1}^n \alpha_i,\ \alpha_i \in N_0 \equiv \{0,1,2,\cdots\}. \tag{2}$$

The boundary operator $\mathcal{B}$ is also a partial differential operator of order at most $2m-1$, given by

$$\begin{aligned} \mathcal{B}\varphi &= \{\mathcal{B}_j,\ j = 1,2,\cdots,m\} \\ (\mathcal{B}_j\varphi)(\xi) &\equiv \sum_{|\beta|\le m_j \le 2m-1} b^j_\beta(\xi) D^\beta \varphi, \quad \xi \in \partial\Sigma, \end{aligned} \tag{3}$$