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New representation and factorizations of the higher-order ultraspherical-type differential equations

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ABSTRACT

The paper deals with the class of linear differential equations of any even order $2\alpha+4$, $\alpha \in \mathbb{N}_0$, which are associated with the so-called ultraspherical-type polynomials. These polynomials form an orthogonal system on the interval $[-1,1]$ with respect to the ultraspherical weight function $(1-x^2)^\alpha$ and additional point masses of equal size at the two endpoints. The differential equations of "ultraspherical-type" were developed by R. Koekoek in 1994 by utilizing special function methods. In the present paper, a new and completely elementary representation of these higher-order differential equations is presented. This result is used to deduce the orthogonality relation of the ultraspherical-type polynomials directly from the differential equation property. Moreover, we introduce two types of factorizations of the corresponding differential operators of order $2\alpha+4$ into a product of $\alpha+2$ linear second-order operators.

1. Introduction

In 1994, R. Koekoek [12] discovered a new class of higher-order linear differential equations satisfied by the symmetric "ultraspherical-type" polynomials. These are the particular cases $\alpha = \beta$, $M = N$ of Koornwinder's "Jacobi-type" polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$, $\alpha, \beta > -1$, $M, N \geq 0$, which were introduced in [14] as the orthogonal polynomials with respect to a linear combination of the Jacobi weight function and two "delta functions" at the endpoints of the interval $[-1,1]$, i.e.

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1). \tag{1.1}$$

For any $\alpha \in \mathbb{N}_0$, $M > 0$, it was proved in [12, Theorem 2] that the orthogonal polynomials of ultraspherical-type, $\{P_n^{\alpha,\alpha,M,M}(x)\}_{n=0}^{\infty}$, satisfy a linear differential equation of order $2\alpha+4$, where only the constant term depends on the degree n of the polynomials, namely

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$$M\left\{\sum_{i=2}^{2\alpha+4} c_i^\alpha D_x^i y(x) + c_0^{\alpha,n} y(x)\right\} + \left\{\left(1-x^2\right)D_x^2 y(x) - (2\alpha+2)D_x y(x) + n(n+2\alpha+1)y(x)\right\} = 0. \quad (1.2)$$

Here and in the following, $D_x^i \equiv (D_x)^i$ denotes the i-fold differentiation with respect to x. Furthermore, the constant coefficient reads

$$\begin{aligned} c_0^{\alpha,n} &= 4(2\alpha+3)\binom{n+2\alpha+2}{n-2} \\ &= \frac{4(2\alpha+3)}{(2\alpha+4)!}(n-1)_{2\alpha+4} = \frac{2(n-1)_{2\alpha+4}}{(\alpha+2)(2\alpha+2)!}, \end{aligned} \quad (1.3)$$

and the coefficient functions $c_i^\alpha(x)$, $i = 2, \cdots, 2\alpha+4$, are given by

$$c_i^\alpha(x) = (2\alpha+3)\left(1-x^2\right)\frac{2^i}{i!}\sum_{k=\max(0,i-\alpha-3)}^{i-2}\binom{\alpha+1}{i-2-k}\frac{(2\alpha+5-i)_k}{k!}\left(\frac{x-1}{2}\right)^k. \quad (1.4)$$

Notice that the coefficient function of the highest derivative simplifies to

$$\begin{aligned} c_{2\alpha+4}^\alpha(x) &= (2\alpha+3)\left(1-x^2\right)\frac{2^{2\alpha+4}}{(2\alpha+4)!}\sum_{k=\alpha+1}^{2\alpha+2}\binom{\alpha+1}{2\alpha+2-k}\frac{(1)_k}{k!}\left(\frac{x-1}{2}\right)^k \\ &= \frac{2^{2\alpha+3}}{(\alpha+2)(2\alpha+2)!}\left(1-x^2\right)\left(\frac{x-1}{2}\right)^{\alpha+1}\sum_{k=0}^{\alpha+1}\binom{\alpha+1}{k}\left(\frac{x-1}{2}\right)^k \\ &= -\frac{2^{2\alpha+3}}{(\alpha+2)(2\alpha+2)!}\left(x^2-1\right)\left(\frac{x-1}{2}\right)^{\alpha+1}\left(\frac{x+1}{2}\right)^{\alpha+1} \\ &= -\frac{2}{(\alpha+2)(2\alpha+2)!}\left(x^2-1\right)^{\alpha+2}. \end{aligned} \quad (1.5)$$

As M tends to $0+$, the remaining part of Eq. (1.2) yields the classical second-order ultraspherical equation (with parameter $\alpha \in \mathbb{N}_0$)

$$\begin{aligned} \mathbf{L}_{2,x}^\alpha y(x) &:= \left[\left(x^2-1\right)D_x^2 + (2\alpha+2)xD_x\right]y(x) \\ &= \left(x^2-1\right)^{-\alpha}D_x\left[\left(x^2-1\right)^{\alpha+1}D_x y(x)\right] = n(n+2\alpha+1)y(x). \end{aligned} \quad (1.6)$$

The bounded eigensolutions of this singular differential equation are the ultraspherical polynomials of order $n \in \mathbb{N}_0$ with well-known hypergeometric representation

$$P_n^{\alpha,\alpha}(x) = \frac{(\alpha+1)_n}{n!}R_n^{\alpha,\alpha}(x), \quad R_n^{\alpha,\alpha}(x) = {}_2F_1\left(-n, n+2\alpha+1; \alpha+1; \frac{1-x}{2}\right). \quad (1.7)$$

For any property of special functions used in this paper we refer to [1] or [6,7].

If $\alpha = 0$, Eq. (1.2) reduces to the so-called Legendre-type equation

$$\begin{aligned} &-\frac{1}{2}\left(1-x^2\right)^2 y^{(4)}(x) + 4Mx\left(1-x^2\right)y^{(3)}(x) \\ &\quad + (6M+1)\left(1-x^2\right)y''(x) - 2xy'(x) + \frac{1}{2}n(n+1)\left[(n-1)(n+2)M+2\right]y(x) = 0. \end{aligned} \quad (1.8)$$

This equation was discovered by H.L. Krall in his pioneering work dating back to 1938/1940 when he characterized all orthogonal polynomial systems which satisfy a linear differential equation of fourth order, see [16,17] as well as A.M. Krall [15]. Later on, L.L. Littlejohn [18] introduced and investigated the so-called