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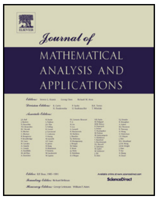

Existence of frequency modes coupling seismic waves and vibrating tall buildings

Darko Volkov [*], Sergey Zheltukhin

Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, United States

A R T I C L E   I N F O

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A B S T R A C T

We prove in this paper an existence result for frequency modes coupling seismic waves and vibrating tall buildings. The derivation from physical principles of a set of equations modeling this phenomenon was done in previous studies. In this model all vibrations are assumed to be anti-plane and time harmonic so the two dimensional Helmholtz equation can be used. A coupling frequency mode is obtained once we can determine a wavenumber such that the solution of the corresponding Helmholtz equation in the lower half plane with relevant Neumann and Dirichlet conditions at the interface satisfies a specific integral equation at the base of an idealized tall building. Although numerical simulations suggest that such wavenumbers should exist, as far as we know, to date, there is no theoretical proof of their existence. This is what this present study offers to provide.

1. Introduction

The traditional approach to evaluating seismic risk in urban areas is to consider seismic waves under ground as the only cause for motion above ground. In earlier studies, seismic wave propagation was evaluated in an initial step and in a second step impacts on man made structures were inferred. However, observational evidence has since then suggested that when an earthquake strikes a large city, seismic activity may in turn be altered by the response of the buildings. This phenomenon is referred to as the "city-effect" and has been studied by many authors, see [8,2].

More recently, in [4], Ghergu and Ionescu have derived a model for the city effect based on the equations of solid mechanics and appropriate coupling of the different elements involved in the physical set-up of the problem. They then proposed a clever way to compute a numerical solution to their system of equations. In this way, in [4], Ghergu and Ionescu were able to compute a city frequency constant: given the geometry and the specific physical constants of an idealized two dimensional city, they computed a frequency that leads to coupling between vibrating buildings and underground seismic waves. In this present paper our goal is

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[*] Corresponding author.
E-mail addresses: darko@wpi.edu (D. Volkov), sergey@wpi.edu (S. Zheltukhin).

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to prove that the equations modeling the city effect introduced in [4] are solvable. We acknowledge that these equations were carefully derived following the laws of solid mechanics combined with the knowledge of the relevant dominant effects causing this phenomenon. There is also ample numerical evidence that these coupling frequencies should exist, see [4,14], at least in the range of physical parameters under consideration in these numerical simulations. As far as we know, there is no mathematical proof, however, that these coupling frequencies must exist. To fill this gap, we will show in this paper that given a building with positive height, mass, and elastic modulus, the (rather involved) set of coupled equations given in [4] defining frequencies coupling that building and the ground beneath have at least one solution, if some constants coming from non-dimensionalization of physical parameters satisfy a sign condition. Furthermore, we show that once the constants for the physical properties of the ground and the building are fixed, the set of all possible coupling frequencies is finite.

Here is an outline of this paper. In Section 2 we introduce the equations defining frequency modes coupling seismic waves and vibrating tall buildings. For the sake of brevity we directly provide them in non-dimensional form. A derivation from physical principles and non-dimensionalization calculations can be found in [4,14]. In Section 2 we introduce on the one hand the PDE modeling the propagation of time harmonic waves under ground while at the ground level a building is subject to a given displacement and a no force condition is applied elsewhere, and on the other hand the integral equation ensuring coupling between vibrations under ground and in the building. The set of coupled equations for which we prove existence is comprised of the PDE and of the coupling integral equation.

In Section 3 we have to carefully study the boundary Dirichlet to Neumann operator T_k for Helmholtz problems outside the unit disk in $\mathbb{R}^2$, where k is the wavenumber. We are aware that this is a well known operator, however, for our purposes, we need to show the lesser known fact that T_k is (strongly) real analytic in k, and we need to determine the strong limit of T_k as k tends to zero. In Section 4 we reformulate our half plane problem to the whole plane using a symmetry: that way the operator T_k introduced in Section 3 can be used. Since the strong limit of T_k was found in Section 3, it is then possible to study the low frequency behavior of our problem thanks to manipulations of k-dependent variational problems.

Section 5 deals with the much more delicate question of high frequency asymptotics. Understanding how waves behave at high frequencies has always piqued the interest of scientists. The geometric optics approximation has been known for quite some time; in the late 19th century Kirchhoff wrote down specific equations capturing the behavior of waves at high frequencies. A rigorous mathematical study of these phenomena first appeared in papers by Majda, Melrose and Taylor, see [10–12]. We note, however, that their results are limited to the case where scatterers are convex domains, so their results not applicable to our particular case. More recently, Chandler-Wilde, Hewett, and Langdon, see [5,6], published continuity and coercivity estimates pertaining to either scattering in dimension 2 by soft or hard line segments (our case), or scattering in dimension 3 by soft or hard open planar surfaces. These estimates include bounds whose **explicit dependence on wavenumbers** is stated and proved. In Section 5, after first informally deriving the expected behavior of some quantities relevant to the coupling frequency problem, we turn to the rigorous proof of that expected result. This is where the new estimates by Chandler-Wilde et al. turn out to be crucial. Finally, in Section 6, we combine all the intermediate results obtained in previous sections to complete the proof of our main theorem. In Section 7, there ensues a brief discussion on our findings and on how we plan to extend this present study to more complex geometries in future work. This paper also contains Appendix A with an overview of results on Hankel functions relevant to our work.

2. The equations defining frequency modes coupling seismic waves to vibrating tall buildings. Statement of main theorem

Following [4] and [14] we model the ground to be the elastic half-space $x_2 > 0$ in three dimensional space, where (x_1, x_2, x_3) is the space variable. We only considered the **anti-plane shearing** case: all displacements