J. Math. Anal. Appl. 421 (2015) 314–328

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic behavior of the partial derivatives of Laguerre kernels and some applications

Francisco Marcellán [a,b], M. Francisca Pérez-Valero [b,*], Yamilet Quintana [c]

[a] *Instituto de Ciencias Matemáticas (ICMAT), Campus de Cantoblanco, Universidad Autónoma de Madrid, Calle Nicolás Cabrera, no. 13-15, 28049 Madrid, Spain*
[b] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain*
[c] *Departamento de Matemáticas Puras y Aplicadas, Universidad Simón Bolívar, Edificio Matemáticas y Sistemas (MYS), Apartado Postal: 89000, Caracas 1080 A, Venezuela*

ARTICLE INFO

Article history:
Received 21 March 2014
Available online 11 July 2014
Submitted by P. Nevai

Keywords:
Orthogonal polynomials
Laguerre polynomials
Asymptotics
Diagonal Christoffel–Darboux kernels
Sobolev-type orthogonal polynomials

ABSTRACT

The aim of this paper is to present some new results about the asymptotic behavior of the partial derivatives of the kernel polynomials associated with the Gamma distribution. We also show how these results can be used in order to obtain the inner relative asymptotics for certain Laguerre–Sobolev type polynomials.

1. Introduction

Let μ be a finite positive Borel measure supported on an infinite subset of $\mathbb{R}$. It is well-known that the polynomial kernels (also called reproducing, Christoffel–Darboux or Dirichlet kernels) associated with the sequences of orthogonal polynomials corresponding to μ are frequently used as a basic tool in spectral analysis, convergence of orthogonal expansions [2,23,27], and other aspects of mathematical analysis (see [26] and the references therein). In the setting of orthogonal polynomial theory these kernels have been especially used by Freud and Nevai [4,21,22] and, more recently, the remarkable Lubinsky's works [9,10] have caused heightened interest in this topic. Also, other interesting and related results corresponding to Fourier–Sobolev expansions may be found in [11–15,17–19,25].

* Corresponding author.
E-mail addresses: pacomarc@ing.uc3m.es (F. Marcellán), mpvalero@math.uc3m.es (M.F. Pérez-Valero), yquintana@usb.ve (Y. Quintana).

http://dx.doi.org/10.1016/j.jmaa.2014.07.012
0022-247X/

Our goal here will be to analyze the asymptotic behavior of the partial derivatives of the diagonal Christoffel–Darboux kernels corresponding to classical Laguerre orthogonal polynomials (in short, the diagonal Laguerre kernels), i.e., we will consider the n-th Christoffel–Darboux kernel $K_n(x,y)$, given by

$$K_n(x,y) = \sum_{k=0}^{n} \frac{\widehat{L}_k^\alpha(x)\widehat{L}_k^\alpha(y)}{\langle \widehat{L}_k^\alpha, \widehat{L}_k^\alpha\rangle_\alpha},$$

and its partial derivatives

$$K_n^{(j,k)}(x,y) := \frac{\partial^{j+k} K_n(x,y)}{\partial x^j \partial y^k}, \quad 0 \le i,j \le n,$$

where, as it is usual, $\{\widehat{L}_n^\alpha(x)\}_{n\ge 0}$ is the sequence of monic polynomials orthogonal with respect to the inner product

$$\langle f,g\rangle_\alpha = \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx, \quad \alpha > -1,\ f,g \in \mathbb{P},$$

and $\mathbb{P}$ denotes the linear space of polynomials with real coefficients. Then, for $c > 0$ we will study the asymptotic behavior of $K_n^{(j,k)}(c,c)$, $0 \le j,k \le n$. From this starting point, we will focus our attention on the study of asymptotic properties of the sequences of polynomials orthogonal with respect to the following Sobolev-type inner product on the linear space of polynomials with real coefficients $\mathbb{P}$:

$$\langle f,g\rangle_S = \langle f,g\rangle_\alpha + \sum_{k=0}^{N} M_k f^{(k)}(c) g^{(k)}(c), \tag{1}$$

where $c > 0$, $M_k \ge 0$, for $k = 0,\dots,N-1$, and $M_N > 0$.

To the best of our knowledge, asymptotic properties of the diagonal Laguerre kernels $K_n^{(j,k)}(c,c)$, $0 \le j,k \le n$, are not available in the literature up to those cases where the following situations have been considered.

- *Case 1:* $c \ge 0$ and either $j = k = 0$ or $0 \le j,k \le 1$ (cf. [6,8]).
- *Case 2:* $c = 0$ and either $0 \le j,k \le 1$ or $0 \le j,k \le n$ (cf. [3,24]).

The outline of the paper is as follows. Section 2 provides some basic background about structural and asymptotic properties of the classical Laguerre polynomials. The estimates of the partial derivatives of the diagonal Christoffel–Darboux kernels $K_n^{(j,k)}(c,c)$ (Theorem 2) are deduced. In Section 3 we prove our main result (Theorem 3), where an estimate in the Laguerre weighted L^2-norm for the difference between Laguerre orthonormal polynomials and the Laguerre–Sobolev type polynomials $\widetilde{L}_n^{\alpha,\underline{M}}(x)$, orthogonal with respect to (1) with $c > 0$, is obtained.

Let consider the multi-indexes $\underline{M} = (M_0,\dots,M_N)$ of nonnegative real numbers, $M_N > 0$. The notation $u_n \sim_n v_n$ will always mean that the sequence u_n/v_n converges to 1 when n tends to infinity. Positive constants will be denoted by C, $C_1, C_{i,j},\dots$ and they may vary at every occurrence. Any other standard notation will be properly introduced whenever needed.

2. Asymptotics for the partial derivatives of the diagonal Laguerre kernels

For $\alpha > -1$, let $\{\widehat{L}_n^\alpha(x)\}_{n\ge0}$, $\{\tilde{L}_n^\alpha(x)\}_{n\ge0}$, and $\{L_n^{(\alpha)}(x)\}_{n\ge0}$ be the sequences of monic, orthonormal and normalized Laguerre polynomials with leading coefficient equal to $\frac{(-1)^n}{n!}$, respectively. The following