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Schatten class Toeplitz operators on generalized Fock spaces

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ABSTRACT

In this paper we characterize the Schatten p class membership of Toeplitz operators with positive measure symbols acting on generalized Fock spaces for the full range $0 < p < \infty$.

1. Introduction

Let $d^c = \frac{i}{4}(\overline{\partial} - \partial)$ and let d be the usual exterior derivative. Throughout the paper, let $\phi \in C^2(\mathbb{C}^n)$ be a real valued function on $\mathbb{C}^n$ such that

$$c\omega_0 < dd^c\phi < C\omega_0 \tag{1.1}$$

holds uniformly pointwise on $\mathbb{C}^n$ for some positive constants c and C (in the sense of positive $(1,1)$ forms) where $\omega_0 = dd^c|\cdot|^2$ is the standard Euclidean Kähler form.

Define F^2_ϕ to be the set of entire functions such that

$$\int_{\mathbb{C}^n} \left|f(z)\right|^2 e^{-2\phi(z)} dv(z) < \infty.$$

Denote by P the orthogonal projection of $L^2(e^{-2\phi}dv)$ onto F^2_ϕ. For a positive measure μ, define the Toeplitz operator $T_\mu : F^2_\phi \to F^2_\phi$ with symbol μ by setting

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$$T_\mu f(z) = \int\limits_{\mathbb{C}^n} K(z,w) f(w) e^{-2\phi(w)} d\mu(w),$$

where K stands for the reproducing (or Bergman) kernel of F^2_ϕ, that is,

$$K(z,w) = \sum_{k=1}^{\infty} f_k(z)\overline{f_k(w)},$$

where $\{f_k\}$ is any orthonormal basis for F^2_ϕ. In the next section we list some recent results on the reproducing kernel (see [7]), which will be crucial to the proofs of our main results on Schatten class properties of Toeplitz operators.

In [3,6] (see also a recent monograph of Zhu [9]), Toeplitz and Hankel operators were considered in the setting of the standard weighted Fock spaces, that is, when $\phi(z) = \frac{\alpha}{2}|z|^2$ for $\alpha > 0$. In [3] characterizations of bounded, compact and Schatten class Toeplitz operators with positive measure symbols were provided (moreover, see [7] for a similar characterization of bounded and compact Toeplitz operators with positive measure symbols on F^2_ϕ). In particular, the Schatten class membership of these Toeplitz operators was characterized in terms of the heat (Berezin) transform of the symbol and in terms of the averaging function $\mu(B(\cdot, r))$. In [6] the boundedness and compactness of Hankel operators on the standard weighted Fock spaces were characterized in terms of BMO and VMO, respectively.

In this paper we will provide very similar characterizations of the Schatten class membership of these Toeplitz operators. Note that unlike the classical Fock space setting where one can utilize explicit formulas for the reproducing kernel, we instead must rely on some known estimates on the behavior of the reproducing kernel (see the first three lemmas in the next section). The proofs of our characterizations will (as usual) be divided into the two cases $0 < p \leq 1$ (which will be dealt with in Section 2) and $p > 1$ (which will be dealt with in Section 3).

Let us note that one can easily write the so called "Fock–Sobolev spaces" from [1] as a weighted Fock space F^2_ϕ with ϕ satisfying (1.1), so that in particular our results immediately apply to these spaces (see [2] for more details).

Finally, note that we will often use the notation $A \lesssim B$ for two nonnegative quantities A and B if $A \leq CB$ for an unimportant constant C. Moreover, the notation $A \gtrsim B$ and $A \approx B$ will have similar meanings.

2. The case $0 < p \leq 1$

In this section we will characterize the Schatten p class T_μ for the case $0 < p \leq 1$. We will often use the following three lemmas from [7].

Lemma 2.1. *If K is the reproducing kernel of F^2_ϕ then there exists $\epsilon_0 > 0$ where*

$$e^{-\phi(w)}\left|K(z,w)\right|e^{-\phi(z)} \lesssim e^{-\epsilon_0|z-w|}.$$

Lemma 2.2. *There exists $\delta > 0$ where*

$$e^{-\phi(w)}\left|K(z,w)\right|e^{-\phi(z)} \gtrsim 1$$

for all $w \in B(z,\delta)$. In particular, $K(z,z)e^{-2\phi(z)} \approx 1$.

Lemma 2.3. *If $r > 0$ then there exists $C_r > 0$ independent of $f \in F^2_\phi$ where*

$$\left|f(z)e^{-\phi(z)}\right|^2 \lesssim C_r \int\limits_{B(z,r)} \left|f(w)e^{-\phi(w)}\right|^2 dv(w).$$