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# Disjointly homogeneous rearrangement invariant spaces via interpolation

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ABSTRACT

A Banach lattice $E$ is called *p-disjointly homogeneous*, $1 \leq p \leq \infty$, when every sequence of pairwise disjoint normalized elements in $E$ has a subsequence equivalent to the unit vector basis of $\ell_p$. Employing methods from interpolation theory, we clarify which r.i. spaces on $[0,1]$ are $p$-disjointly homogeneous. In particular, for every $1 < p < \infty$ and any increasing concave function $\varphi$ on $[0,1]$, which is not equivalent to neither 1 nor $t$, there exists a $p$-disjointly homogeneous r.i. space with the fundamental function $\varphi$. Moreover, it is shown that given $1 < p < \infty$ and an increasing concave function $\varphi$ with non-trivial dilation indices, there is a unique $p$-disjointly homogeneous space among all interpolation spaces between the Lorentz and Marcinkiewicz spaces associated with $\varphi$.



## 1. Introduction

A Banach lattice $E$ is called *disjointly homogeneous* if any two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ of pairwise disjoint normalized elements from $E$ contain subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$, respectively, which are equivalent in $E$. Similarly, $E$ will be called *p-disjointly homogeneous*, $1 \leq p \leq \infty$, when every sequence of pairwise disjoint normalized elements in $E$ has a subsequence equivalent to the unit vector basis of $\ell_p$. The importance of disjointly homogeneous spaces introduced in [16] is based, mainly, on close connections between the classes of compact and strictly singular operators acting in such spaces. Recall that a linear operator between Banach spaces is *strictly singular* if it is not an isomorphism when restricted to any infinite-dimensional subspace. Recently, it was obtained a series of interesting results showing that a strictly singular operator has some compact power whenever the Banach lattice in which it is bounded is disjointly homogeneous (see [14–17]); in particular, in the paper [14], it is proved that every strictly singular operator in a $p$-disjointly homogeneous rearrangement invariant (r.i.) space with lower Boyd index $\alpha_X > 0$ has compact square and if $p = 2$, such an operator even is compact itself. For this reason it is important to

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know how wide is the class of disjointly homogeneous Banach lattices. As is shown in the above cited papers, it contains $L_p(\mu)$-spaces, $1 \le p \le \infty$, Lorentz function spaces $L_{q,p}$ and $\Lambda(W,p)$, certain classes of Orlicz function spaces and also some discrete spaces such as the Tsirelson space.

The main aim of this paper is to clarify which r.i. spaces on $[0,1]$ are $p$-disjointly homogeneous. We focused on the more interesting reflexive case, when $1 < p < \infty$. Our approach to this problem is based on using tools from interpolation theory, especially, the real and complex methods of interpolation. By the complex method of interpolation, we prove that for every $1 < p < \infty$ and any increasing concave function $\varphi$ on $[0,1]$, which is not equivalent to neither 1 nor $t$, there exists a $p$-disjointly homogeneous r.i. space with the fundamental function $\varphi$ (Corollary 1). Note that there is the only r.i. space on $[0,1]$, $L_\infty$ (resp. $L_1$), having the fundamental function equivalent to 1 (resp. $t$). This result is new even for the power functions $\varphi(t) = t^\alpha$, $0 < \alpha < 1$. Moreover, it is shown that given $1 < p < \infty$ and an increasing concave function $\varphi$ with non-trivial dilation indices, there is a unique $p$-disjointly homogeneous space among all interpolation spaces between the Lorentz and Marcinkiewicz spaces associated with $\varphi$ (namely, the Lorentz space $\Lambda_{p,\varphi}$ with the quasi-norm $\|x\|_{p,\varphi} = (\int_0^1 [x^*(t)\varphi(t)]^p \, dt/t)^{1/p}$, see Theorem 2). At the same time, in Section 4, for every $1 < p < \infty$ and any increasing concave function $\varphi$ on $[0,1]$ such that $\lim_{t\to 0} \varphi(t) = 0$ and upper dilation index $\beta_\varphi < 1$, we construct a $p$-disjointly homogeneous r.i. space with the fundamental function $\varphi$, which is not interpolation with respect to the corresponding couple of Lorentz and Marcinkiewicz spaces (Theorem 3). Finally, in Section 5, we investigate some properties of sequences of pairwise disjoint functions in the real interpolation spaces $(X_0, X_1)_{\theta,p}$ ($0 < \theta < 1$, $1 \le p < \infty$) provided that $X_1 \subset X_0$.

## 2. Preliminaries

### 2.1. Rearrangement invariant spaces

In this subsection we present some definitions and auxiliary results from the theory of rearrangement invariant spaces. For more details on that theory we refer to [8,19,21].

A Banach function space $X = (X, \|\cdot\|)$ of (classes of) real measurable functions $x(t)$ defined on the interval $[0,1]$ is said to be *rearrangement invariant (r.i.) space* if the conditions $x^*(t) \le y^*(t)$ a.e. on $[0,1]$ and $y \in X$ imply $x \in X$ and $\|x\|_X \le \|y\|_X$. Here, $x^*$ denotes the non-increasing right-continuous rearrangement of $|x(s)|$ given by

$$x^*(t) = \inf\{\tau \ge 0 : m(\{s \in [0,1] : |x(s)| > \tau\}) \le t\}, \quad 0 \le t \le 1,$$

where $m$ is the Lebesgue measure.

For every r.i. space $X$ on $[0,1]$ we have the continuous embeddings $L_\infty[0,1] \subset X \subset L_1[0,1]$. The *fundamental function* of an r.i. space $X$ is given by $\varphi_X(t) := \|\chi_a\|_X$, $m(a) = t$, $0 \le t \le 1$, where $\chi_a$ denotes the characteristic function of a measurable set $a \subset [0,1]$. It is well known that every fundamental function is quasi-concave on $[0,1]$, i.e., it is non-decreasing and the function $\varphi_X(t)/t$ is non-increasing on $(0,1]$. Each quasi-concave function $\varphi$ on $[0,1]$ is equivalent to its least concave majorant $\bar{\varphi}$, more exactly, $\frac{1}{2}\bar{\varphi}(t) \le \varphi(t) \le \bar{\varphi}(t)$ $(0 \le t \le 1)$ [19, Theorem 2.1.1].

If $X$ is an r.i. space on $[0,1]$, then *the Köthe dual* space $X'$ consists of all measurable functions $y$ such that

$$\|y\|_{X'} = \sup\left\{\int_0^1 x(t)y(t)\,dt : \|x\|_X \le 1\right\} < \infty.$$

The space $X'$ is r.i. as well; it is embedded into the dual space $X^*$ of $X$ isometrically, and $X' = X^*$ if and only if $X$ is separable. An r.i. space $X$ is said *to have the Fatou property* if the conditions $x_n \in X$