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Parameter convexity and concavity of generalized trigonometric functions

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ABSTRACT

We study the convexity properties of the generalized trigonometric functions viewed as functions of the parameter. We show that $p \to \sin_p(y)$ and $p \to \cos_p(y)$ are log-concave on the appropriate intervals while $p \to \tan_p(y)$ is log-convex. We also prove similar facts about the generalized hyperbolic functions. In particular, our results settle a major part of the conjecture recently put forward in [4].

1. Introduction and preliminaries

The symbol $\mathbb{R}_+$ will mean $[0, \infty)$. There are several ways in the literature to define generalized trigonometric functions (see, for instance, [11,12,17,18,24]). We will stick with the definition adopted in the book [16]. For $p > 0$ define a differentiable function $F_p : [0, 1) \to \mathbb{R}_+$ by

$$F_p(x) = \int_0^x \left(1 - t^p\right)^{-1/p} dt. \tag{1}$$

Clearly, $F_2 = \arcsin$ so that F_p can be viewed as generalized arcsine $F_p(x) = \arcsin_p(x)$. Since F_p is strictly increasing it has an inverse denoted by $\sin_p$. In all the references we could find the range of p is restricted to $(1, \infty)$ because only in this case $\sin_p(x)$ can be made periodic like usual sine. Nothing prohibits, however, defining $\sin_p(x)$ for all $p > 0$, so we will be dealing with such generalized case here. If $p > 1$ the function $\sin_p(x)$ is defined on the interval $[0, \pi_p/2]$, where

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$$\pi_p = 2\int_0^1 \left(1-t^p\right)^{-1/p} dt = \frac{2\pi}{p\sin(\pi/p)}.$$

It is convenient to extend the above definition by setting $\pi_p = +\infty$ for $0 < p \le 1$. We will adopt this convention throughout the paper. In this way the function $y \to \sin_p(y)$ is strictly increasing on $[0, \pi_p/2]$ with $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$ in analogy with the usual sine. It is easily seen that $p \to \pi_p$ is strictly decreasing on $(1, \infty)$ and maps this interval onto itself. For $p > 1$ the definition is extended to $[0, \pi_p]$ by

$$\sin_p(y) = \sin(\pi_p - y) \quad \text{for } y \in [\pi_p/2, \pi_p];$$

further extension to $[-\pi_p, \pi_p]$ is made by oddness; finally $\sin_p$ is extended to the whole $\mathbb{R}$ by $2\pi_p$ periodicity. If $p \in (0, 1]$ the inverse of $F_p(x)$ is defined on $\mathbb{R}_+$ and we just need oddness to extend the definition to the whole real line. The limiting cases are (see also [8]):

$$\sin_0(y) = 0 \quad \text{on } \mathbb{R}, \qquad \sin_1(y) = 1 - e^{-y} \quad \text{on } \mathbb{R}_+, \qquad \sin_\infty(y) = y \quad \text{on } [0, 1]. \tag{2}$$

Since

$$\frac{d}{dy}\sin_p(y) = \left(\frac{dF_p(x)}{dx}\right)^{-1}_{|x=\sin_p(y)} = \left(1 - \left[\sin_p(y)\right]^p\right)^{1/p}, \tag{3}$$

we get $\sin_p'(0) = 1$ and $\sin_p'(\pi_p/2) = 0$, which shows that $\sin_p(y)$ is continuously differentiable on $\mathbb{R}$ for all $p > 0$. The continuous derivative above is naturally called the generalized cosine:

$$\cos_p(y) = \frac{d}{dy}\sin_p(y), \quad y \in \mathbb{R}. \tag{4}$$

When $y \in [0, \pi_p/2]$ (for $p > 1$) and $y \in \mathbb{R}_+$ (for $0 < p \le 1$) we can also define $\cos_p(y)$ by the right hand side of (3) which leads to an integral representation for $\arccos_p$:

$$\cos_p(y) = x = \left(1 - \left[\sin_p(y)\right]^p\right)^{1/p} \quad \Rightarrow \quad y = \arcsin_p\left(\left(1-x^p\right)^{1/p}\right) = \int_0^{(1-x^p)^{\frac{1}{p}}} \frac{dt}{(1-t^p)^{1/p}},$$

or, by substitution $s = (1-t^p)^{1/p}$,

$$y = \arccos_p(x) = \int_0^{(1-x^p)^{\frac{1}{p}}} \frac{dt}{(1-t^p)^{1/p}} = \int_x^1 \frac{s^{p-2}ds}{(1-s^p)^{1-\frac{1}{p}}}, \quad 0 \le x \le 1. \tag{5}$$

The function $\cos_p$ can now be defined on $[0, \pi_p/2]$ as the inverse function to $\arccos_p$ and extended to $\mathbb{R}$ by evenness and $2\pi_p$ periodicity. The limiting values for $p = 0, 1, \infty$ can be obtained by differentiating (2). Pursuing an analogy with trigonometric functions further, the generalized tangent function is defined by

$$\tan_p(y) = \frac{\sin_p(y)}{\cos_p(y)}, \tag{6}$$

where $y \in \mathbb{R}\backslash\{(\mathbb{Z}+1/2)\pi_p\}$ if $p > 1$. If $0 < p \le 1$ the function $\tan_p(y)$ is continuous on $\mathbb{R}$. It is easy to show by differentiation that $\tan_p(y)$ is the inverse function to

$$\arctan_p(x) = \int_0^x \frac{dt}{1+t^p}, \quad 0 \le x < \infty, \tag{7}$$