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On traveling wave solutions of the θ-equation of dispersive type

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ABSTRACT

Traveling wave solutions to a class of dispersive models,

$$u_t - u_{txx} + uu_x = \theta uu_{xxx} + (1-\theta)u_x u_{xx},$$

are investigated in terms of the parameter θ, including two integrable equations, the Camassa–Holm equation, $\theta = 1/3$, and the Degasperis–Procesi equation, $\theta = 1/4$, as special models. It was proved in H. Liu and Z. Yin (2011) [39] that when $1/2 < \theta \leq 1$ smooth solutions persist for all time, and when $0 \leq \theta \leq \frac{1}{2}$, strong solutions of the θ-equation may blow up in finite time, yielding rich traveling wave patterns. This work therefore restricts to only the range $\theta \in [0, 1/2]$. It is shown that when $\theta = 0$, only periodic travel wave is permissible, and when $\theta = 1/2$ traveling waves may be solitary, periodic or kink-like waves. For $0 < \theta < 1/2$, traveling waves such as periodic, solitary, peakon, peaked periodic, cusped periodic, or cusped soliton are all permissible.

1. Introduction

In this work, we investigate traveling wave solutions to a class of dispersive models – the θ-equation of the form [37]

$$\left(1-\partial_x^2\right)u_t + \left(1-\theta\partial_x^2\right)\left(\frac{u^2}{2}\right)_x = (1-4\theta)\left(\frac{u_x^2}{2}\right)_x, \quad x \in \mathbb{R},\ t > 0. \tag{1.1}$$

The equation can be formally rewritten as

$$u_t - u_{txx} + uu_x = \theta uu_{xxx} + (1-\theta)u_x u_{xx}, \tag{1.2}$$

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which when $0 < \theta < 1$ involves a convex combination of nonlinear terms uu_{xxx} and $u_x u_{xx}$. In (1.1), two equations are worth of special attention: $\theta = \frac{1}{3}$ and $\theta = \frac{1}{4}$. The θ-equation when $\theta = \frac{1}{3}$ reduces to the Camassa–Holm (CH) equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, in which $u(x,t)$ denotes the fluid velocity at time t in the spatial x direction [3,20,32]. The CH equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [13,15]. Taking $\theta = \frac{1}{4}$ in (1.1) one finds the Degasperis–Procesi (DP) equation [16]. The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as that for the CH equation [21,22]. This θ-class may well have more applications than those mentioned here.

In recent years, nonlocal dispersive models as an extension of the classical KdV equation have been investigated intensively at different levels of treatments: modeling, analysis as well as numerical simulation. The model derives in several ways, for instance, (i) the asymptotic modeling of shallow water waves [45,21,22]; (ii) renormalization of dispersive operators [45,36]; and (iii) model equations of some dispersive schemes [37]. The peculiar feature of nonlocal dispersive models is their capability to capture both smooth long wave propagation and short wave breaking phenomena. The study of traveling wave solutions to dispersive equations proves to be insightful in understanding various wave structures involved in the dispersive wave dynamics, we refer to recent works [19,29,33,34,40,46,51] for such investigations.

1.1. θ-Equations

The class of θ-equations was identified by H. Liu [37] in the study of model equations for some dispersive schemes to approximate the Hopf equation

$$u_t + uu_x = 0.$$

With $\theta = \frac{1}{b+1}$ the model (1.1) under a transformation as shown in [39] links to the b-model,

$$u_t - \alpha^2 u_{txx} + c_0 u_x + (b+1)uu_x + \Gamma u_{xxx} = \alpha^2(bu_x u_{xx} + uu_{xxx}),$$

which has been extensively studied in recent years [18,16,24,25,30,31]. Both classes of equations are contained in the more general class introduced in [36] using renormalization of dispersive operators and number of conservation laws, so called the B-equations

$$u_t + uu_x + \big[Q * B(u, u_x)\big]_x = 0, \tag{1.3}$$

where $Q = \frac{1}{2}e^{-|x|}$ and B is a quadratic function of u and u_x. For this class the local well-posedness in $C([0,T); H^{3/2+}(\mathbb{R})) \cap C^1([0,T); H^{1/2+}(\mathbb{R}))$ for (1.3) with initial data $u_0 \in H^{3/2+}$ is shown in [36]. In fact, up to a scaling of $t \to \frac{t}{\theta}$ for $\theta \neq 0$, the θ-equation can indeed be rewritten as (1.3) with

$$B = \left(\frac{1}{\theta} - 1\right)\frac{u^2}{2} + \left(4 - \frac{1}{\theta}\right)\frac{u_x^2}{2}.$$

In the last decades, a lot of analysis has been given to the CH equation and the DP equation, among other dispersive equations.

The CH equation has a bi-Hamiltonian structure [28,35] and is completely integrable [3,7]. Its solitary waves are smooth if $c_0 > 0$ and peaked in the limiting case $c_0 = 0$ [4]. The orbital stability of the peaked solitons is proved in [12], and that of the smooth solitons in [14]. The explicit interaction of the peaked solitons is given in [1]. It has been shown that the Cauchy problem of the CH equation is locally well-posed [8,43] for initial data $u_0 \in H^{3/2+}(\mathbb{R})$. Moreover, it has global strong solutions [6,8] and also admits finite time blow-up solutions [6,8,9]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [2,10,11,47].