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A blowing up wave equation with exponential type nonlinearity and arbitrary positive energy

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ABSTRACT

We consider in two space dimensions, the nonexistence of global solutions to the class of Klein–Gordon equations

$$\partial_{tt}u - \Delta u + u = f_a(u), \quad (t,x) \in \left[0, T^*\right) \times \Omega.$$

We give sufficient conditions on the data such that the solution of the above Klein–Gordon equation blows up in finite time.

1. Introduction

This paper studies the nonexistence of global solutions to the semilinear wave equation

$$\begin{cases} \partial_{tt}u - \Delta u + u = f_a(u) & \text{on } [0,T) \times \Omega; \\ (u, \partial_t u)_{|t=0} = (u_0, u_1), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth domain, a is a real parameter and $u = u(t,x)$ is a real valued function. The above wave problem has various applications in the area of nonlinear optics, plasma physics, fluid mechanics, etc. [23].

Before going further, we recall a few historic facts about this problem. We begin with the defocusing semi-linear wave equation with power p in space dimensions $d \geq 3$,

$$\partial_{tt}u - \Delta u + |u|^{p-1}u = 0, \quad p > 1. \tag{1.2}$$

The well-posedness of (1.2) in the scale of the Sobolev spaces H^s has been widely investigated, see for instance [3,4,6,10,21,22]. It is well-known that the Cauchy problem associated to (1.2) is locally well-posed

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in the usual Sobolev space $H^s(\mathbb{R}^d)$ if $s > \frac{d}{2}$, or when $\frac{1}{2} \le s < \frac{d}{2}$ and $p \le 1 + \frac{4}{d-2s}$. Moreover, if $p = 1 + \frac{4}{d-2s}$ and $\frac{1}{2} \le s < \frac{d}{2}$, then we have global H^s-solutions for small Cauchy data [17].

The global solvability in the energy space has a long history. A critical value of the power p appears, namely $p_c := \frac{d+2}{d-2}$, and there are mainly three cases. In the subcritical case ($p < p_c$), Ginibre and Velo proved in [4] the global existence and uniqueness in the energy space.

In the critical case ($p = p_c$), the global existence was first proved by Struwe in the radially symmetric case [24], then by Grillakis [5] in the general case and later on by Shatah and Struwe [22] in other dimensions.

In the supercritical case ($p > p_c$), the question remains open except for some partial results [12,13].

In two space dimensions, any polynomial nonlinearity is subcritical with respect to the H^1 norm. So it's legitimate to consider an exponential nonlinearity. In fact, Nakamura and Ozawa [17] proved global well-posedness and scattering for small Cauchy data in any space dimension $d \ge 2$ (see also [18]). Later on, A. Atallah Baraket [1] showed a local existence result to the 2D equation

$$(E_\alpha) \quad \partial_{tt} u - \Delta_x u + u e^{\alpha u^2} = 0$$

for $0 < \alpha < 4\pi$ and with radially symmetric initial data $(0, u_1)$ having compact support. Ibrahim, Majdoub and Masmoudi [8] obtained global well-posedness in the energy space when the data belongs to the unit ball of energy (see [7] for similar result in the case of a bounded domain). Recently, Struwe [25,26] proved unconditional existence of global solution with regular datum.

In [15], the authors proved unconditional global well-posedness and linearization of the defocusing semi-linear wave equation (1.1) for $f_a(u) = -(e^u - 1) + u$. The same result was proved in [16] for a class of exponential types of nonlinearities. (Similar results were proved in the case of Schrödinger equation [20].)

Results about the blow-up properties for the local solution of (1.1) with $f_a(u) = u + u|u|^{p-1}$ in dimension larger than two were obtained in [2,14,19,23,28]. The blow-up problem was treated [11] with a new point of view which consists of comparing the energy of the data to the energy of the ground state. This work was generalized recently in [9] for more types of nonlinearities. In these both results the energy of the ground state is an upper bound of the initial energy. It is the aim of this paper to obtain a blowing up result within any upper bound of the initial energy, extending [27]. Using previous works [15,16], we obtain a blowing up result for arbitrary positive initial energy in two space dimensions with exponential type nonlinearity.

The nonlinearity considered in this paper and its primitive vanishing on zero are

$$f_a(u) := e^u - 1 + au \quad \text{and} \quad F_a(u) := \int_0^u f_a(s)\, ds = e^u - 1 - u + \frac{a}{2} u^2.$$

The Cauchy problem (1.1) was studied in [15,16], where a local well-posedness result was proved in the energy space. Moreover, the solution satisfies conservation of the energy

$$E(t) := E\big(u(t)\big) := \frac{1}{2}\big(\|u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)}\big) - \int_\Omega F_a\big(u(t,x)\big)\, dx.$$

Here and hereafter, for any real number ε and any $u \in H^1(\Omega)$, we define the quantities

$$G(t) := \|u(t)\|^2_{L^2(\Omega)};$$

$$I(t) := \|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} - \int_\Omega u(t,x) f_a\big(u(t,x)\big)\, dx;$$

$$J_\varepsilon(t) := \|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} - (2+\varepsilon) \int_\Omega F_a\big(u(t,x)\big)\, dx.$$