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Asymptotic formulae for generalized Freud polynomials ☆

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ABSTRACT

We establish the Mehler–Heine type formulae for orthonormal polynomials with respect to generalized Freud weights. Using this type of asymptotics, we can give estimates of the value at the origin of these polynomials and of all their derivatives as well as the asymptotic behavior of the corresponding zeros.

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1. Introduction

The theory of orthogonal polynomials is a major topic in Approximation Theory. Since the nineteenth century they have been studied widely, especially when they are orthogonal with respect to a standard inner product, i.e., an inner product $(\cdot,\cdot)$ defined in a pre-Hilbert space H containing the space of the polynomials $\mathbb{P}$ such that $(xf,g) = (f,xg)$, for all $f,g \in H$. A very important case of this type of inner products refers to the ones generated by means of weight functions. Thus, if we consider $W(x)$ a weight function on an interval $I \subseteq \mathbb{R}$, then we can construct an inner product as

$$(f,g) = \int_I f(x)g(x)W(x)dx, \quad \text{where } f,g \in L^2_W := \left\{ f : \int_I f^2(x)W(x)dx < \infty \right\}.$$

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In this paper, we consider the exponential weights $W_\alpha(x) = \exp(-c|x|^\alpha)$, $\alpha > 1$, on the real line where $c > 0$ is a normalization constant. The orthogonal polynomials with respect to the inner product

$$(f, g) = \int_{\mathbb{R}} f(x)g(x)W_\alpha^2(x)dx,$$

are so-called Freud orthogonal polynomials. The literature on this topic has been very wide since the sixties when G. Freud started to study these weights, though it is mandatory to cite two very nice and deep books: one by A.L. Levin and D.S. Lubinsky [6] and the other one by E.B. Saff and V. Totik [9]. About the asymptotics of the corresponding orthogonal polynomials, the first results correspond to the cases $\alpha = 4$ with $c = 1/2$ [8] and $\alpha = 6$ with $c = 1/12$ [10]. Later, using the powerful Riemann–Hilbert method, several authors have given precise asymptotic results (see the survey [13] and the references therein).

From now on, we choose $c = 1$, i.e.

$$W_\alpha(x) = \exp\bigl(-|x|^\alpha\bigr), \quad \alpha > 1. \tag{1}$$

We denote by $(p_n)_n$ the sequence of orthonormal polynomials with respect to $W_\alpha^2(x)$, $p_n(x) = \gamma_n x^n + \dots$, with $\gamma_n > 0$. These weights can be generalized considering $W_{\alpha,m}(x) = x^m \exp(-|x|^\alpha)$ with $\alpha > 1$ and $m \in \mathbb{N} \cup \{0\}$. Thus, the functions

$$W_{\alpha,m}^2(x) = x^{2m} \exp\bigl(-2|x|^\alpha\bigr), \quad \alpha > 1,\ m \in \mathbb{N} \cup \{0\}, \tag{2}$$

are weights on the real line. We denote by $(p_n^{[m]})_n$ the sequence of orthonormal polynomials with respect to (2), $p_n^{[m]}(x) = \gamma_n^{[m]} x^n + \dots$, with $\gamma_n^{[m]} > 0$. Clearly, when $m = 0$ we have the Freud polynomials, i.e., $p_n^{[0]} = p_n$, for all n. Thus, the polynomials $p_n^{[m]}$ are so-called *generalized Freud orthonormal polynomials*. These polynomials belong to a wider class of weights on the real line given by $x^{2m} \exp(-Q(x))$ with Q belonging to the class $\mathcal{F}(C^2+)$ (see [6] to get more information about this and other classes). In more general frameworks, asymptotic properties of these polynomials have been obtained, for example, in [6] or in [14] using powerful techniques.

The main aim of this paper is to establish the Mehler–Heine type asymptotics of the sequence $(p_n^{[m]})_n$, and as an immediate consequence we deduce the asymptotic behavior of the corresponding zeros. Besides, this formula also permits to obtain asymptotic estimates of $(p_n^{[m]})^{(j)}(0)$ with $j = 0, \dots, n$.

The structure of the paper is the following. In Section 2, we establish the Mehler–Heine type formulae for the sequence $(p_n)_n$ on compact subsets of the complex plane. In addition, we obtain estimates for $(p_n)^{(j)}(0)$, $j = 0, \dots, n$, when $n \to \infty$. In Section 3, we give our main result about the Mehler–Heine asymptotics of the generalized Freud orthonormal polynomials and their consequences on the asymptotic behavior of $(p_n^{[m]})^{(j)}(0)$ and on the zeros of this family of orthogonal polynomials.

Throughout the paper we use the notation $x_n \simeq y_n$, when $n \to \infty$, meaning $\lim_{n\to\infty} x_n/y_n = 1$.

2. Freud orthonormal polynomials

As we have commented in the Introduction, in this section we will establish the Mehler–Heine type asymptotics of the polynomials p_n. Thus, we have

Theorem 1. *Let $(p_n)_n$ be the sequence of Freud orthonormal polynomials with respect to the weight function $W_\alpha^2(x)$ defined by (1). Then, the polynomials p_n satisfy the following Mehler–Heine type formulae*

$$\lim_{n\to\infty} (-1)^n a_{2n}^{1/2} p_{2n}\left(\frac{z}{b_{2n}}\right) = \sqrt{\frac{2}{\pi}} \cos z,$$

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