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Bootstrapping and Askey–Wilson polynomials ☆

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ABSTRACT

The mixed moments for the Askey–Wilson polynomials are found using a bootstrapping method and connection coefficients. A similar bootstrapping idea on generating functions gives a new Askey–Wilson generating function. Modified generating functions of orthogonal polynomials are shown to generate polynomials satisfying recurrences of known degree greater than three. An important special case of this hierarchy is a polynomial which satisfies a four term recurrence, and its combinatorics is studied.

1. Introduction

The Askey–Wilson polynomials [1] $p_n(x; a, b, c, d|q)$ are orthogonal polynomials in x which depend upon five parameters: a, b, c, d and q. In [2, §2] Berg and Ismail use a bootstrapping method to prove orthogonality of Askey–Wilson polynomials by initially starting with the orthogonality of the $a = b = c = d = 0$ case, the continuous q-Hermite polynomials, and successively proving more general orthogonality relations, adding parameters along the way.

In this paper we implement this idea in two different ways. First, using successive connection coefficients for two sets of orthogonal polynomials, we will find explicit formulas for generalized moments of Askey–Wilson polynomials, see Theorem 2.4. This method also gives a heuristic for a relation between the two measures of the two polynomial sets, see Remark 2.3, which is correct for the Askey–Wilson hierarchy. Using this idea we give a new generating function (Theorem 2.9) for Askey–Wilson polynomials when $d = 0$.

The second approach is to assume the two sets of polynomials have generating functions which are closely related, up to a q-exponential factor. We prove in Theorem 3.1, Theorem 3.6, and Theorem 3.15 that if one set is an orthogonal set, the second set has a recurrence relation of predictable order, which may be greater than three. We give several examples using the Askey–Wilson hierarchy.

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Finally we consider a more detailed example of the second approach, using a generating function to define a set of polynomials called the discrete big q-Hermite polynomials. These polynomials satisfy a 4-term recurrence relation. We give the moments for the pair of measures for their orthogonality relations. Some of the combinatorics for these polynomials are given in Section 5. Finally we record in Proposition 6.1 a possible q-analogue of the Hermite polynomial addition theorem.

We shall use basic hypergeometric notation, which is in Gasper and Rahman [5] and Ismail [6].

2. Askey–Wilson polynomials and connection coefficients

The connection coefficients are defined as the constants obtained when one expands one set of polynomials in terms of another set of polynomials.

For the Askey–Wilson polynomials [6, 15.2.5, p. 383]

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad)_n}{a^n} {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \,\middle|\, q;q\right), \quad x=\cos\theta$$

we shall use the connection coefficients obtained by successively adding a parameter

$$(a,b,c,d) = (0,0,0,0) \to (a,0,0,0) \to (a,b,0,0) \to (a,b,c,0) \to (a,b,c,d).$$

Using a simple general result on orthogonal polynomials, we derive an almost immediate proof of an explicit formula for the mixed moments of Askey–Wilson polynomials.

First we set the notation for an orthogonal polynomial set $p_n(x)$. Let $\mathcal{L}_p$ be the linear functional on polynomials for which orthogonality holds

$$\mathcal{L}_p\bigl(p_m(x)p_n(x)\bigr) = h_n\delta_{mn}, \quad 0 \le m,n.$$

Definition 2.1. The mixed moments of $\mathcal{L}_p$ are $\mathcal{L}_p(x^n p_m(x))$, $0 \le m,n$.

The main tool is the following proposition, which allows the computation of mixed moments of one set of orthogonal polynomials from another set if the connection coefficients are known.

Proposition 2.2. *Let $R_n(x)$ and $S_n(x)$ be orthogonal polynomials with linear functionals $\mathcal{L}_R$ and $\mathcal{L}_S$, respectively, such that $\mathcal{L}_R(1) = \mathcal{L}_S(1) = 1$. Suppose that the connection coefficients are*

$$R_k(x) = \sum_{i=0}^{k} c_{k,i} S_i(x). \tag{1}$$

Then

$$\mathcal{L}_S\bigl(x^n S_m(x)\bigr) = \sum_{k=0}^{n} \frac{\mathcal{L}_R(x^n R_k(x))}{\mathcal{L}_R(R_k(x)^2)} c_{k,m} \mathcal{L}_S\bigl(S_m(x)^2\bigr).$$

Proof. If we multiply both sides of (1) by $S_m(x)$ and apply $\mathcal{L}_S$, we have

$$\mathcal{L}_S\bigl(R_k(x)S_m(x)\bigr) = c_{k,m}\mathcal{L}_S\bigl(S_m(x)^2\bigr).$$

Then by expanding x^n in terms of $R_k(x)$

$$x^n = \sum_{k=0}^{n} \frac{\mathcal{L}_R(x^n R_k(x))}{\mathcal{L}_R(R_k(x)^2)} R_k(x)$$