J. Math. Anal. Appl. 421 (2015) 779–805

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On discrete symplectic systems: Associated maximal and minimal linear relations and nonhomogeneous problems

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ARTICLE INFO

Article history:
Received 28 June 2013
Available online 11 July 2014
Submitted by D. O'Regan

Keywords:
Discrete symplectic system
Time-reversed system
Definiteness condition
Nonhomogeneous problem
Deficiency index
Linear relation

ABSTRACT

In this paper we characterize the definiteness of the discrete symplectic system, study a nonhomogeneous discrete symplectic system, and introduce the minimal and maximal linear relations associated with these systems. Fundamental properties of the corresponding deficiency indices, including a relationship between the number of square summable solutions and the dimension of the defect subspace, are also derived. Moreover, a sufficient condition for the existence of a densely defined operator associated with the symplectic system is provided.

1. Introduction

The spectral theory for difference equations and systems, which include discrete analogs of Sturm–Liouville and Hamiltonian systems of differential equations, has a long history and a considerable literature which we will not attempt to delineate here other than to cite the following works, and the references therein, to give the reader a sense of the scope of the subject in time and content, cf. [2,7,10–13,34,35,39]. In connection with this, the development of a Weyl–Titchmarsh theory for discrete Hamiltonian and symplectic systems parallel to that which exists for Hamiltonian systems of differential equations is of relatively recent origin, cf. [3–5,9,10,14,26,29,33,36,37]. Our current paper contributes to this ongoing development.

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http://dx.doi.org/10.1016/j.jmaa.2014.07.015

We investigate the nonhomogeneous problem as well as the basic development of linear relations associated with discrete symplectic systems written in the so-called time-reversed form given by

$$z_k(\lambda) = \mathbb{S}_k(\lambda) z_{k+1}(\lambda) \quad \text{with } \mathbb{S}_k(\lambda) := \mathcal{S}_k + \lambda \mathcal{V}_k \text{ and } k \in \mathbb{N}_0, \tag{S$_\lambda$}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, $\mathbb{N}_0 := [0, \infty) \cap \mathbb{Z}$, and $\mathcal{S}_k$ and $\mathcal{V}_k$ are complex $2n \times 2n$ matrices such that

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}, \qquad \mathcal{S}_k^* \mathcal{J} \mathcal{V}_k \text{ is Hermitian}, \qquad \mathcal{V}_k^* \mathcal{J} \mathcal{V}_k = 0, \quad \text{and} \quad \Psi_k := \mathcal{J} \mathcal{S}_k \mathcal{J} \mathcal{V}_k^* \mathcal{J} \geqslant 0, \tag{1.1}$$

where $\mathcal{J}$ represents a $2n \times 2n$ skew-symmetric matrix given by $\mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

We note (viz. Lemma 2.2) that the first, second, and third identities in (1.1), i.e., (1.1)(i)–(iii), can be combined into the single equality

$$\mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} \, \mathbb{S}_k(\lambda) = \mathcal{J} \quad \text{for all } \lambda \in \mathbb{C} \text{ and } k \in \mathbb{N}_0, \tag{1.2}$$

where $\mathbb{S}_k^*(\bar{\lambda}) := (\mathbb{S}_k(\bar{\lambda}))^*$. Identity (1.2) justifies the terminology *symplectic system* for (S$_\lambda$), though system (S$_\lambda$) corresponds to the well-known time-reversed discrete symplectic system introduced in [8, Remark 4] only when $\lambda \in \mathbb{R}$; particularly, the case when $\lambda = 0$. In addition, system (S$_\lambda$) can also be viewed as a perturbation of the original symplectic system $z_k = \mathcal{S}_k z_{k+1}$, i.e., of (S$_\lambda$) with $\lambda = 0$, but for which the fundamental properties of symplectic systems remain true with appropriate, natural, modifications.

In [9,14], the Weyl–Titchmarsh theory was first established for discrete symplectic systems given by

$$z_{k+1}(\lambda) = \mathbb{S}_k^{-1} z_k(\lambda), \quad k \in \mathbb{N}_0, \tag{1.3}$$

in which a special form for $\mathcal{V}_k$ is assumed; the proper generalization is later derived in [37], see also [38]. The results given in [37] for system (1.3) remain valid for system (S$_\lambda$) with standard changes given for the definition of the semi-inner product, viz. (2.13) (cf. [37, Theorem 2.8 and Section 4]), and for the associated weight function, viz. (1.1)(iv) (cf. [37, Identity (1.1)(iv)]).

Consideration here of the time-reversed form given in system (S$_\lambda$), rather than that given in system (1.3), is motivated, in part, by a desire to produce more natural calculations involving the semi-inner product and in particular a more natural form for a Green function associated with nonhomogeneous discrete symplectic systems, viz. Lemma 4.2. We can also associate with system (S$_\lambda$) a densely defined operator, because there is no shift in the associated semi-inner product (cf. Theorem 5.4 and [28]). Moreover, this approach will enable us in subsequent research to generalize these results and unify them with the continuous time case by means of the time scale theory.

Given the inherent semi-definiteness of the function Ψ defined in (1.1) (cf. (2.2)), it is natural to consider the construction of *linear relations* in association with (S$_\lambda$), their extensions and their associated spectral theory. The theory of linear relations provides powerful tools for the study of *multivalued* linear operators in a Hilbert space, especially for non-densely defined linear operators. The study of linear relations in this context traces back to [1]; see also [15–17,20] and the references therein. For linear Hamiltonian differential systems given by

$$-\mathcal{J} z'(t) = \big[H(t) + \lambda W(t)\big] z(t), \tag{1.4}$$

where $H(t)$ and $W(t)$ are Hermitian and $W(t)$ is positive semi-definite, this approach was initiated in [27] and further developed, e.g., in [6,21,22,25].