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Zero sets of bivariate Hermite polynomials ☆

Iván Area [a], Dimitar K. Dimitrov [b,*], Eduardo Godoy [c]

[a] *Departamento de Matemática Aplicada II, E.E. Telecomunicación, Universidade de Vigo, 36310-Vigo, Spain*
[b] *Departamento de Matemática Aplicada, IBILCE, Universidade Estadual Paulista, 15054-000 São José do Rio Preto, SP, Brazil*
[c] *Departamento de Matemática Aplicada II, E.E. Industrial, Universidade de Vigo, 36310-Vigo, Spain*

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ABSTRACT

We establish various properties for the zero sets of three families of bivariate Hermite polynomials. Special emphasis is given to those bivariate orthogonal polynomials introduced by Hermite by means of a Rodrigues type formula related to a general positive definite quadratic form. For this family we prove that the zero set of the polynomial of total degree $n + m$ consists of exactly $n + m$ disjoint branches and possesses $n + m$ asymptotes. A natural extension of the notion of interlacing is introduced and it is proved that the zero sets of the family under discussion obey this property. The results show that the properties of the zero sets, considered as affine algebraic curves in $\mathbb{R}^2$, are completely different for the three families analyzed.

1. Introduction

The properties of the zeros of univariate orthogonal polynomials have been studied thoroughly because of their fundamental role as eigenvalues of Jacobi operators and important applications, such as nodes of Gaussian quadrature formulas. If $\{p_n(x)\}$ is a sequence of orthogonal polynomials on the real line, with respect to a positive Borel measure $d\mu(x)$, it is well known that all zeros of $p_n(x)$ are real, belong to the convex hull of the support of $d\mu(x)$, and are distinct. Moreover, the zeros of two consecutive polynomials $p_n(x)$ and $p_{n+1}(x)$ interlace [8,18].

Despite the growing number of publications on multivariate orthogonal polynomials [4–7,11,20], there are few results on the zero sets of these polynomial families (see [11,14,19,22] and the references therein) and the lack of general results is due to two main reasons. First of all, there is a rich variety of polynomials

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* Corresponding author.
E-mail addresses: area@uvigo.es (I. Area), dimitrov@ibilce.unesp.br (D.K. Dimitrov), egodoy@dma.uvigo.es (E. Godoy).

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of many variables, even if they are orthogonal with respect to a fixed Borel measure on $\mathbb{R}^d$, because of the many possible choices of arrangements of the multivariate polynomials. Moreover, since the zero set of a multivariate polynomial is an algebraic variety, sometimes its study requires deep and fine results from algebraic geometry [16]. The interest on zeros of multivariate polynomials has been focused mainly on describing those families having common zeros, because this property is related to the existence of Gaussian cubature formulas [10,11,13,20–22].

In this paper we are interested in the properties of the zero sets of three families of bivariate Hermite polynomials considered as affine algebraic curves. Let

$$H_n^e(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left[e^{-x^2/2}\right] = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right) \tag{1}$$

be the probabilistic Hermite polynomials, with zeros $h^e_{n,k}$, $k = 1, \dots, n$, and $H_n(x)$ be the Hermite polynomials (see [15, p. 10]):

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left[e^{-x^2}\right], \quad n \geq 0. \tag{2}$$

Consider the general bivariate Hermite polynomials represented as a sum of products of H_n^e in the form (see [3, p. 370, Eq. (21)])

$$\begin{aligned} H_{n,m}(x, y; \Lambda) = {} & \sum_{k=0}^{\min(n,m)} (-1)^k k! \binom{m}{k} \binom{n}{k} a^{(n-k)/2} b^k c^{(m-k)/2} \\ & \times H^e_{n-k}\left(\frac{ax + by}{\sqrt{a}}\right) H^e_{m-k}\left(\frac{bx + cy}{\sqrt{c}}\right), \end{aligned} \tag{3}$$

with $a, c > 0$, $ac - b^2 > 0$, whose orthogonality and basic properties are provided in Section 3. The affine transformation

$$s = \frac{ax + by}{\sqrt{a}}, \qquad t = \frac{bx + cy}{\sqrt{c}} \tag{4}$$

yields

$$\hat{H}_{n,m}(s, t; \Lambda) = \sum_{k=0}^{\min(n,m)} (-1)^k k! \binom{m}{k} \binom{n}{k} a^{(n-k)/2} b^k c^{(m-k)/2} H^e_{n-k}(s) H^e_{m-k}(t). \tag{5}$$

It is clear that the properties of the zero sets of $H_{n,m}(x, y; \Lambda)$ can be easily recovered from those of $\hat{H}_{n,m}(s, t; \Lambda)$ via the transformation inverse to (4). Therefore we state our result for the zero sets (affine algebraic plane curves)

$$Z_{n,m} = \left\{(s, t) \in \mathbb{R}^2 \;\middle|\; \hat{H}_{n,m}(s, t; \Lambda) = 0\right\} \tag{6}$$

of the polynomials $\hat{H}_{n,m}(s, t; \Lambda)$, with $b \neq 0$, as follows:

Theorem 1.1. *The affine algebraic plane curve $Z_{n,m}$ defined in (6):*

- *consists of exactly in $n + m$ disjoint branches;*
- *possesses n vertical asymptotes $s = h^e_{n,k}$, $k = 1, \dots, n$, and m horizontal asymptotes $t = h^e_{m,j}$, $j = 1, \dots m$.*