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The root distribution of polynomials with a three-term recurrence

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ABSTRACT

For any fixed positive integer n, we study the root distribution of a sequence of polynomials $H_m(z)$ satisfying the rational generating function

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^n}$$

where $A(z)$ and $B(z)$ are any polynomials in z with complex coefficients. We show that the roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on a specific fixed real algebraic curve for all large m.

1. Introduction

The sequence of polynomials $H_m(z)$, generated by the rational function $1/(1 + B(z)t + A(z)t^n)$, has the three-term recurrence relation of degree n

$$H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-n}(z) = 0 \tag{1}$$

and the initial conditions

$$H_m(z) = (-1)^m B^m(z), \quad 0 \leq m < n. \tag{2}$$

For the study of the root distribution of other sequences of polynomials that satisfy three-term recurrences, see [8,14]. In [16], the author shows that in the three special cases when $n = 2$, 3, and 4, the roots of $H_m(z)$ which satisfy $A(z) \neq 0$ will lie on the curve $\mathcal{C}$ defined in Theorem 1, and are dense there as $m \to \infty$. This paper shows that for any fixed integer n, this result holds for all large m in the theorem below.

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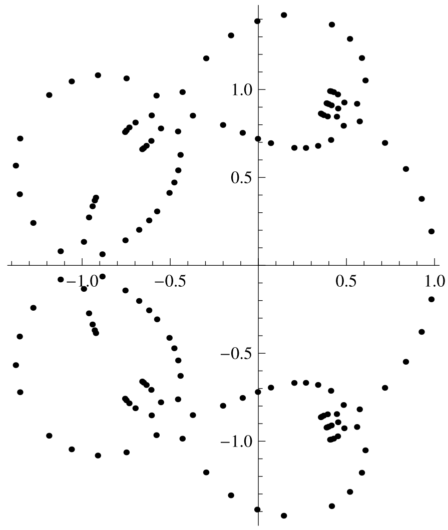

Fig. 1. Distribution of the quotients of roots of the hexic denominator.

Theorem 1. *Let $H_m(z)$ be a sequence of polynomials whose generating function is*

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^n}$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients. There is a constant $C = C(n)$ such that for all $m > C$, the roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on a fixed curve $\mathcal{C}$ given by

$$\Im \frac{B^n(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq (-1)^n \Re \frac{B^n(z)}{A(z)} \leq \frac{n^n}{(n-1)^{n-1}}$$

and are dense there as $m \to \infty$.

This theorem holds when the numerator of the generating function is a monomial in t and z. For a general numerator, it appears, in an unpublished joint work with Robert Boyer, that the set of roots will approach $\mathcal{C}$ and a possible finite set in the Hausdorff metric on the non-empty compact subsets. For more study of sequences of polynomials whose roots approach fixed curves, see [6,7]. Other studies of the limits of zeros of polynomials satisfying a linear homogeneous recursion whose coefficients are polynomials in z are given in [4,5].

An important trinomial is $y^m - my + m - 1$. Its fundamental role in the study of inequalities is pointed out in both [3] and [13]. This paper shows that a "θ-analogue" of this ($\theta = 0$) trinomial is fundamental for the study of polynomials generated by rational functions whose denominators are trinomials. Some further information about trinomials is available in [9]. It is also noteworthy that although there is no really concise formula for the discriminant of a general polynomial in terms of its coefficients, there is such a formula for the discriminant of a trinomial [11, pp. 406–407]. Here we develop, in the fashion of Ismail, a q-analogue of this discriminant formula. This plays a fundamental role in the determination of the curve $\mathcal{C}$.

Our main approach is to count the number of roots of $H_m(z)$ on the curve $\mathcal{C}$ and show that this number equals the degree of this polynomial. This number of roots connects with the number of quotients of roots in t of the denominator $1 + B(z)t + A(z)t^n$ on a portion of the unit circle. The plot of these quotients when $n = 6$ and $m = 30$ is given in Fig. 1. Although the curve $\mathcal{C}$ depends on $A(z)$ and $B(z)$, it will be seen that this plot of the quotients is independent of these two polynomials.