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Prescribed diagonal Ricci tensor in locally conformally flat manifolds

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ABSTRACT

In the Euclidean space $(\mathbb{R}^n, g)$, with $n \geq 3$, $g_{ij} = \delta_{ij}$, we consider a diagonal $(0,2)$-tensor $T = \sum_i f_i(x) dx_i^2$. We obtain necessary and sufficient conditions for the existence of a metric $\bar{g}$, conformal to g, such that $\mathrm{Ric}_{\bar{g}} = T$, where $\mathrm{Ric}_{\bar{g}}$ is the Ricci curvature tensor of the metric $\bar{g}$. The solution to this problem is given explicitly for special cases of the tensor T, including singular tensors and cases where the metric $\bar{g}$ is complete on $\mathbb{R}^n$. Similar problems are considered for locally conformally flat manifolds.

1. Introduction

In [9] Milnor considered the problem of understanding Ricci curvature as a fundamental problem of present-day mathematics. One basic question is to determine which symmetric covariant tensors of rank two can be Ricci tensors of Riemannian metrics. Therefore we formulate the following problem:

(P) Given a symmetric $(0,2)$-tensor T, defined on a manifold M^n, $n \geq 3$, does there exist a Riemannian metric g, such that $\mathrm{Ric}_g = T$?

Studying problem (P) corresponds to solving a system of nonlinear second-order differential equations. There are the same number of equations as unknowns in the system because g and T are both symmetric $n \times n$ matrices. However there is a complicating factor since, according to [5], any solution of $\mathrm{Ric}_g = T$ must also satisfy the Bianchi identity given by

$$\mathrm{Bian}(g,T) = g^{ab}\left(T_{am;b} - \frac{1}{2}T_{ab;m}\right) = 0.$$

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DeTurck showed in [4] that, whenever $n \geq 3$, problem (P) admits a local solution when the given tensor T is nonsingular. Moreover, he presented examples of singular tensors T for which there is not a metric g satisfying $\mathrm{Ric}_g = T$ even locally. When T is singular, but still has constant rank and satisfies certain appropriate conditions, the Ricci equation also admits local solutions (see [6]). Once the existence of local solutions for the problem (P) when the given tensor T is non-singular has been established, other aspects of the problem (P) can be considered such as the existence and uniqueness of global solutions and completeness.

Results on the existence and uniqueness of solutions to problem (P) when M^n is a two-dimensional manifold can be found in [2] and [5]. For compact manifolds some nonexistence results can be found in [7] and [8]. Cao and DeTurck in [3] studied the existence of global solutions on balls (including all of $\mathbb{R}^n$) and spheres for rotationally symmetric nonsingular tensors. For such tensors on balls, they showed that problem (P) has a unique (up to homothety) local rotationally symmetric solution, and they studied the question of existence of complete metrics of this type. On the sphere S^n, they proved some nonexistence results, and they obtained necessary conditions on a rotationally symmetric tensor T for the existence of a metric g satisfying $\mathrm{Ric}_g = T$ on S^n. Cao and DeTurck, in [3] also studied the system $\mathrm{Ric}_g = T$ in $\mathbb{R}^n$ for rotationally symmetric tensors and proved that a metric g satisfying $\mathrm{Ric}_g = T$ must be conformally flat.

Pina and Tenenblat have obtained results for the problem (P) for special classes of tensors T and conformal metrics (see [10] and [12] and their references).

In this paper, we consider a diagonal $(0, 2)$-tensor T on the Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, and provide necessary as well as sufficient conditions, for the existence of a metric conformal to g, whose Ricci tensor is the given tensor T. Moreover, we extend the theory to locally conformally flat manifolds.

More precisely, in the Euclidean space $(\mathbb{R}^n, g)$, with $n \geq 3$, coordinates $x = (x_1, .., x_n)$ and $g_{ij} = \delta_{ij}$ we consider the diagonal $(0, 2)$-tensor $T = \sum_i f_i(x)dx_i^2$, where each $f_i(x)$ is a smooth function. For such tensors, we want to find a smooth positive function φ so that the Ricci tensor of the metric $\bar{g} = \frac{1}{\varphi^2}g$ is T.

As a consequence of these results we exhibit examples of tensors T for which there exists a complete metric $\bar{g}$, conformal to the Euclidean metric, such that $\mathrm{Ric}_{\bar{g}} = T$, including a case where T is singular.

2. Preliminaries

In the Euclidean space $(\mathbb{R}^n, g)$, $n \geq 3$, with coordinates $x = (x_1, .., x_n)$ and $g_{ij} = \delta_{ij}$, we consider a diagonal $(0, 2)$-tensor $T = \sum_i f_i(x)dx_i^2$, with $f_i(x)$ smooth functions. We seek necessary and sufficient conditions on the tensor T for the existence of a metric $\bar{g} = \frac{1}{\varphi^2}g$ such that $\mathrm{Ric}_{\bar{g}} = T$. That is, given the tensor T, we want to solve the problem

$$\begin{cases} \bar{g} = \dfrac{1}{\varphi^2} g, \\ \mathrm{Ric}_{\bar{g}} = T \end{cases} \tag{2.1}$$

for the function φ.

As $\bar{g}$ is conformal to the Euclidean metric g, the Ricci tensor of $\bar{g}$ is given by

$$\mathrm{Ric}_{\bar{g}} = \frac{1}{\varphi^2}\left\{(n-2)\varphi \operatorname{Hess}_g \varphi + \left(\varphi \Delta_g \varphi - (n-1)\|\nabla_g \varphi\|^2\right) g\right\} \tag{2.2}$$

and the scalar curvature of $\bar{g}$ is given by

$$\bar{K} = (n-1)\left(2\varphi \Delta_g \varphi - n\|\nabla_g \varphi\|^2\right), \tag{2.3}$$

where $\operatorname{Hess}_g \varphi$ is the Hessian (matrix of second partial derivatives) of the function φ with respect to the Euclidean metric, and Δ_g and ∇_g denote the Laplacian and the gradient in the Euclidean metric g, respectively (see [1]).