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Note

A note on monolithic scattered compacta

Jesús Ferrer [1]

Departamento de Anàlisis Matemático, Universidad de Valencia, Dr. Moliner, 50, 46100 Burjasot (Valencia), Spain

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ABSTRACT

For a Banach space E, it is well-known that a necessary condition for E to have the controlled separable complementation property ($CSCP$, for short) is that the dual unit ball B_{E^*} be monolithic in the weak-star topology. We prove here that when X is a scattered first countable locally compact space, then monolithicity of X turns out to be sufficient for $C_0(X)$ to enjoy the $CSCP$.

1. Introduction

There are a number of different complementation properties of great importance to study the geometrical structure of a Banach space, see for instance [10–12]. In these last years we have focused our attention in the study of a particular one of these properties, namely the controlled separable complementation property ($CSCP$), see [3,6,7].

For the sake of completeness, let us just say that a Banach space E (always real) is said to possess the $CSCP$ if, for every two separable subspaces U and V of E and E^*, respectively, there is a bounded projection P on E such that

(i) $P(E)$ is separable,
(ii) $U \subseteq P(E)$,
(iii) $V \subseteq P^*(E^*)$.

In [3], motivated by having observed that the space $C[0,\omega_1]$ has the $CSCP$, we started studying this property for $C(K)$ spaces when K is the one-point compactification of a locally compact scattered topological space. Later on, in [7] and [6], we noticed that in general a necessary condition for a space $C(K)$ to have

E-mail address: Jesus.Ferrer@uv.es.

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the *CSCP* is that the compact K be monolithic, even this topological property turned out to be sufficient when K is a Mrówka compact, that is, a particular type of one-point compactification which is a scattered compact of height 3. Ever since, we wondered if this last result could admit a generalization to a broader class of scattered compacta. In [5] we proved that if K is a scattered compact of finite height, say $n+1$, and each point of $K \setminus K^{(n)}$ has a countable neighborhood base, then $C(K)$ has the *CSCP*. This result clearly extends the analogous one for the Mrówka compacta, although it does not apply to the space $C[0, \omega_1]$ since the scattered compact $[0, \omega_1]$ has uncountable height. Needless to say, this is mainly why we kept seeking for a result of this type for arbitrary scattered compacta.

The notion of monolithic space is due to Arkhangel'skii, [2]. A topological space X is said to be monolithic whenever each separable subset is second countable. For X being locally compact, by Uryshon's metrization theorem, X is monolithic whenever each separable subset is metrizable. Translated to scattered locally compact spaces, this means that in order to be monolithic every countable subset must have countable closure. One of the best known classes of this type of spaces is the one formed by Corson compacta. Nevertheless, stepping into a more general setting, let us mention that the classes of Valdivia and monolithic compacta are not comparable: The space $\{0, 1\}^{\omega_1}$ is Valdivia but, since it is separable and not metrizable, it is not monolithic, while the Mrówka compact $K_{\mathcal{L}}$, where $\mathcal{L}$ is a non-trivial *ladder system* on ω_1, see [6], is monolithic but it is not even a continuous image of a Valdivia compact, otherwise since it is scattered of finite height, it would be an Eberlein compact, see [9], but in [6] we proved this not to be so.

In [1], under *CH*, a compact L was constructed such that it is Corson non-metrizable and carries a strictly positive Radon measure of countable type and $C(L)$ is not Weakly Lindelöf Determined (*WLD*). Notice that each *WLD* space has the *CSCP*, see for instance [8]. The compact L is clearly monolithic. Besides, since it carries a strictly positive measure of countable type, it belongs to a class of compacta which we introduced in [3], and studied in [4], under the name of *countably measure determined* compacta. For the sake of completeness, let us just say that a compact K is countably measure determined whenever there is a countable set $\{\mu_j : \ j \geq 1\}$ of Radon measures in K such that, considered as elements of the dual space $C(K)^*$, $\bigcap_{j\geq 1} ker\, \mu_j = \{0\}$; notice that this is equivalent to saying that $C(K)^*$ is weak-star separable. In [3], we showed that there is a necessary condition slightly stronger than K being monolithic for $C(K)$ to have the *CSCP*, namely that each countably measure determined closed subset of K must be metrizable. Hence, given that the compact L does not satisfy this requirement, we have that $C(L)$ not only is not *WLD* but it does not even have the *CSCP*. Therefore, under *CH*, the condition of K being monolithic is not sufficient in general to guarantee that $C(K)$ has the *CSCP*. Let us mention that, in comparison with the well known result stating that a Banach space E is *WLD* if and only if the dual unit ball B_{E^*} is a Corson compactum for the weak-star topology, the question of whether the weak-star monolithicity of the dual unit ball B_{E^*} is sufficient for the space E to have the *CSCP* seems to still be open.

2. The *CSCP* and monolithic scattered compacta

In what follows X will be a scattered locally compact topological space of Cantor–Bendixson height η, i.e., η is the smallest ordinal such that $X^{(\eta)} = \emptyset$. We shall use the following notation, for $\alpha < \eta$,

$$M_\alpha := X^{(\alpha)} \setminus X^{(\alpha+1)},$$

where $X^{(0)} = X$. The sets M_α are clearly pairwise disjoint and

$$X = \bigcup_{\alpha<\eta} M_\alpha.$$

In the following we give some auxiliary lemmas leading to show that if X is first countable and monolithic then the space $C_0(X)$, i.e., the Banach space of continuous real-valued functions vanishing at infinity, has