J. Math. Anal. Appl. 421 (2015) 1081–1100

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Linearity of the volume. Looking for a characterization of sausages ☆

Eugenia Saorín Gómez [a,*], Jesús Yepes Nicolás [b]

[a] *Institut für Algebra und Geometrie, Otto-von-Guericke Universität Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Germany*
[b] *Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain*

ARTICLE INFO

Article history:
Received 3 February 2014
Available online 30 July 2014
Submitted by J. Bastero

Keywords:
Linearity of volume
Sausages
Brunn–Minkowski inequality
Roots of Steiner polynomial
Inner parallel body
Linearity of determinant

ABSTRACT

Let K, E, L be convex bodies, $\dim L \leq 1$ and $K = L + E$, a sausage. In this case $\mathrm{vol}(\lambda K + (1-\lambda)E) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E)$. We prove that under the sole assumption that K and E have an equal volume projection (or a common maximal volume section), if the above equality holds for just one value in $(0, 1)$, then $K = L + E$ with $\dim L \leq 1$. However, even having equality for all $\lambda \in [0, 1]$, if no extra assumption on K, E is done, such a characterization is not possible. This problem is connected with a conjecture relating the roots of the Steiner polynomial of a pair of convex bodies to their relative inradius. Counterexamples for the general case are explicitly given. In the same line, a counterexample to a conjecture by Matheron on inner parallel bodies is also shown.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., non-empty compact convex sets, not necessarily with non-empty interior, in the n-dimensional Euclidean space $\mathbb{R}^n$. The n-dimensional volume of a measurable set $M \subsetneq \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (or $\mathrm{vol}_n(M)$ if the distinction of the dimension is useful). With $\dim M$, $\mathrm{conv}\, M$ and $\mathrm{aff}\, M$ we represent its dimension (namely, the dimension of its affine hull), convex hull and affine hull, respectively. We denote by B_n the n-dimensional Euclidean unit ball, by $\mathbb{S}^{n-1}$ its boundary and, in particular, we write $\kappa_n = \mathrm{vol}(B_n)$. Finally, the set of all k-dimensional (linear) planes of $\mathbb{R}^n$ is denoted by $\mathcal{L}^n_k$, and for $H \in \mathcal{L}^n_k$, $K \in \mathcal{K}^n$, the orthogonal projection of K onto H is denoted by $K|H$ and with $H^\perp \in \mathcal{L}^n_{n-k}$ we represent the orthogonal complement of H.

The volume of a positive linear combination of two convex bodies $K, E \in \mathcal{K}^n$, $\mu, \lambda \geq 0$, is a homogeneous polynomial of degree n in μ and λ, namely

☆ First author is supported by Dirección General de Investigación MTM2011-25377 MCIT and FEDER. Second author is supported by MINECO project MTM2012-34037. Both of them are also supported by "Programa de Ayudas a Grupos de Excelencia de la Región de Murcia", Fundación Séneca, 04540/GERM/06.

* Corresponding author.
E-mail addresses: eugenia.saorin@ovgu.de (E. Saorín Gómez), jesus.yepes@um.es (J. Yepes Nicolás).

http://dx.doi.org/10.1016/j.jmaa.2014.07.060
0022-247X/

$$\mathrm{vol}(\mu K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\mu^{n-i}\lambda^i. \tag{1.1}$$

The coefficients $\mathrm{W}_i(K;E)$ are the *relative quermassintegrals* of K with respect to E, and they are a special case of the more general defined *mixed volumes* for which we refer to [17, Section 5.1]. In particular, we have $\mathrm{W}_0(K;E) = \mathrm{vol}(K)$, $\mathrm{W}_n(K;E) = \mathrm{vol}(E)$ and $\mathrm{W}_i(K;E) = \mathrm{W}_{n-i}(E;K)$. If $\mu = 1$ in (1.1) then the above expression is known as the (relative) *Steiner polynomial* or *Steiner formula* of K (with respect to E).

Let us consider $\mathrm{V}_{K;E}(\lambda) = \mathrm{vol}(\lambda K + (1-\lambda)E)$ the volume of the convex combination of $K, E \in \mathcal{K}^n$ for $\lambda \in [0,1]$. From (1.1) follows that $\mathrm{V}_{K;E}(\lambda)$ is a polynomial of degree (at most) n, namely,

$$\mathrm{V}_{K;E}(\lambda) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^{n-i}(1-\lambda)^i.$$

Brunn–Minkowski's inequality (for an extensive and beautiful survey on this inequality we refer to [8]) ensures that the function $\mathrm{V}_{K;E}^{1/n}$ defined on $\lambda \in [0,1]$ is concave. It is known that under special assumptions on the convex bodies K, E ([4, Section 50], [16], [9, Subsection 1.2.4], [13]) the classic Brunn–Minkowski inequality can be refined obtaining that

$$\mathrm{V}_{K;E}(\lambda) \geq \lambda\,\mathrm{vol}(K) + (1-\lambda)\,\mathrm{vol}(E).$$

The first goal along this paper is to understand/characterize the (pairs of) convex bodies K, E for which there is equality in this inequality; i.e., for which the polynomial $\mathrm{V}_{K;E}$ has degree one. In this case, we would have

$$\mathrm{V}_{K;E}(\lambda) = \lambda\,\mathrm{vol}(K) + (1-\lambda)\,\mathrm{vol}(E), \tag{1.2}$$

and we will say that $\mathrm{V}_{K;E}$ is linear in $\lambda \in [0,1]$. From now on, whenever we refer to linearity of the volume we will be meaning (1.2).

The formal polynomial expression in the complex variable z

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)z^i, \tag{1.3}$$

is known as the (relative) Steiner polynomial of K with respect to E. Notice that for $z \geq 0$ it provides the volume of $K + zE$ (cf. (1.1)).

Let $\mathrm{r}(K;E)$ denote the (relative) inradius of K (with respect to E), i.e., $\mathrm{r}(K;E) = \max\{\lambda \geq 0 : x + \lambda E \subset K$ for some $x \in \mathbb{R}^n\}$. For the sake of brevity, we will say that the pair of convex bodies K, E is a sausage if either $K = L + E$ where $L \in \mathcal{K}^n$ with $\dim L \leq 1$ or $E = L + K$ where $L \in \mathcal{K}^n$ with $\dim L \leq 1$.

In [10], the following statement was conjectured

Conjecture 1.1. *Let $K \in \mathcal{K}^n$ with inradius $\mathrm{r}(K;B_n) = 1$. Then -1 is an $(n-1)$-fold root of $f_{K;B_n}$ if and only if K is a sausage with respect to B_n, i.e., $K = L + B_n$ where $L \in \mathcal{K}^n$ with $\dim L \leq 1$.*

Bonnesen's inequality in the plane establishes that

$$\mathrm{W}_0(K;E) - 2\mathrm{W}_1(K;E)\mathrm{r}(K;E) + \mathrm{W}_2(K;E)\mathrm{r}(K;E)^2 \leq 0, \tag{1.4}$$

with equality if and only if $K = L + \mathrm{r}(K;E)E$ with $L \in \mathcal{K}^n$ so that $\dim L \leq 1$ (the inequality was first proved by Bonnesen when $E = B_2$ in [3], and Blaschke generalized it to an arbitrary so-called *gauge* body E in the plane in [2, pp. 33–36]). Thus, Conjecture 1.1 is true in dimension 2 for any gauge body E.