J. Math. Anal. Appl. 421 (2015) 1101–1130

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a family of integrals that extend the Askey–Wilson integral

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ARTICLE INFO

Article history:
Received 15 May 2014
Available online 30 July 2014
Submitted by M.J. Schlosser

Keywords:
Non-uniform lattices
Divided difference operators
Orthogonal polynomials
Semi-classical weights
Askey–Wilson polynomials
Basic hypergeometric integrals

ABSTRACT

We study a family of integrals parameterised by $N = 2, 3, \dots$ generalising the Askey–Wilson integral $N = 2$ which has arisen in the theory of q-analogs of monodromy preserving deformations of linear differential systems and in theory of the Baxter Q operator for the XXZ open quantum spin chain. These integrals are particular examples of moments defined by weights generalising the Askey–Wilson weight and we show the integrals are characterised by various $(N-1)$-th order linear q-difference equations which we construct. In addition we demonstrate that these integrals can be evaluated as a finite sum of $(N-1)$ BC_1-type Jackson integrals or ${}_{2N+2}\varphi_{2N+1}$ basic hypergeometric functions.

1. Introduction

The Askey–Wilson integral occupies an important position in the theory of q-integrals and q-Selberg integrals because the underlying weight stands at the head of the Askey Table of basic and ordinary hypergeometric function orthogonal polynomial systems. We recall the Askey–Wilson weight [1] itself has four parameters $\{a_1, \dots, a_4\}$ with base q

$$w(x) = w\bigl(x; \{a_1, \dots, a_4\}\bigr) = \frac{(z^2, z^{-2}; q)_\infty}{\sin(\theta) \prod_{j=1}^{4} (a_j z, a_j z^{-1}; q)_\infty},$$
$$z = e^{i\theta},\ x = \frac{1}{2}\bigl(z + z^{-1}\bigr) = \cos\theta \in \mathfrak{G} = (-1, 1). \tag{1.1}$$

We also recall that the Askey–Wilson integral is defined by

$$I_2(a_1, a_2, a_3, a_4) := \int_{\mathbb{T}} \frac{dz}{2\pi\sqrt{-1}z} \frac{(z^{\pm 2}; q)_\infty}{\prod_{j=1}^{4} (a_j z^{\pm 1}; q)_\infty},$$

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http://dx.doi.org/10.1016/j.jmaa.2014.07.056

with $|a_j| < 1$ for $j = 1, \dots, 4$ and has the evaluation

$$I_2(a_1, a_2, a_3, a_4) = 2\frac{(a_1a_2a_3a_4;q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(a_ja_k;q)_\infty}.$$

We employ the standard notations for the q Pochhammer symbol, their products, the q-binomial coefficient and the elliptic theta function

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),$$
$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a_1, a_2, a_3, \dots; q)_n = (a_1;q)_n(a_2;q)_n(a_3;q)_n\cdots,$$
$$(az^{\pm 1};q)_\infty = (az, az^{-1};q)_\infty,$$
$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, \quad m = 0, 1, \dots, n \in \mathbb{N},$$
$$\theta(z;q) := (z;q)_\infty(qz^{-1};q)_\infty.$$

Also $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and we will invariably assume that $|q| < 1$, and other conditions on $a_1, a_2, \dots$ to ensure convergence of the infinite products.

Our study treats a family of integrals that extend the Askey–Wilson integral in a direction not considered before. The next member of such a family is (labelled as $N = 3$ whereas the Askey–Wilson integral is labelled by $N = 2$)

$$I_3(a_1, \dots, a_6) := \int_{\mathbb{T}\cup\mathbb{H}} \frac{dz}{2\pi\sqrt{-1}z}\frac{z-z^{-1}}{z^{3/2}-z^{-3/2}}\frac{(z^{\pm 3};q^{3/2})_\infty}{\prod_{j=1}^{6}(a_jz^{\pm 1};q)_\infty}, \tag{1.2}$$

where the contour integral is the sum of a circular integral and a "tail" integral. This integrand possesses sequences of poles which accumulate at either $z = \infty$ and $z = 0$, so the contour $\mathbb{T}$ can separate these two sets if $|a_j| < 1$ and circle the origin in an anti-clockwise direction, as occurs in the Askey–Wilson case. Let us define $a_j = q^{s_j}$, so that in the usual case $|q| < 1$ we require also that $\mathrm{Re}(s_j) > 0$ for $j = 1, \dots, 6$. However this integrand also has a branch cut, conventionally taken to be $(-\infty, 0)$ and so we take the contour path to be of a Hankel type with a tail $\mathbb{H}$ starting at $\infty e^{-\pi i}$ and ending at $\infty e^{+\pi i}$. This contribution involves an integral on $[1, \infty)$ with an integrand whose leading term as $x \to \infty$ is

$$-\frac{1}{\pi}\exp\left(\frac{8\pi^2}{9\log q}\right)q^{\frac{3}{8}+\frac{1}{2}\sum_{j=1}^{6}s_j(s_j-1)}(1-x^{-2})x^{-3+\sum_{j=1}^{6}s_j}\prod_{j=1}^{6}\frac{(-q^{1-s_j}x^{-1};q)_\infty}{(-q^{s_j}x^{-1};q)_\infty},$$

where we have employed the asymptotic expansion of [27], which is valid for $|\arg(a_j)| \le \pi$. This integral clearly converges under the additional condition that $\sum_{j=1}^{6}\mathrm{Re}(s_j) < 2$. Clearly from the prefactors of the above expression the contribution from the tail is exponentially small as $q \to 1^-$.

Our aim, in fact, is to study the generalisations for $N \ge 3$, namely

$$I_N = I_N(a_1, \dots, a_{2N}) := \int_{\mathbb{T}\cup\mathbb{H}} \frac{dz}{2\pi\sqrt{-1}z}\frac{z-z^{-1}}{z^{N/2}-z^{-N/2}}\frac{(z^{\pm N};q^{N/2})_\infty}{\prod_{j=1}^{2N}(a_jz^{\pm 1};q)_\infty}. \tag{1.3}$$