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A dynamic thermoviscoelastic contact problem with the second sound effect ☆

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ABSTRACT

This paper deals with a contact problem describing the mechanical and thermal evolution of a damped extensible thermoviscoelastic beam under the Cattaneo law, relating the heat flux to the gradient of the temperature. The beam is rigidly clamped at its left end whereas the right end of the beam moves vertically between reactive stops like a nonlinear spring. Existence and uniqueness of the solution is proved, as well as the exponential decay of the related energy. Then, fully discrete approximations are introduced by using the classical finite element method and the implicit Euler scheme to approximate the spatial variable and to discretize the time derivatives, respectively. An a priori error estimates result is proved, from which the linear convergence of the algorithm is deduced. The case where the two stops are rigid is also studied from the point of view of the existence and longtime behavior of the solutions. Finally, some numerical simulations are presented to demonstrate the accuracy of the approximation and the behavior of the solution.

1. Introduction

We investigate here the longtime behavior of a nonlinear evolution contact problem describing the vibrations of a damped extensible thermoelastic beam of natural length ℓ. The model is derived by combining the ideas of Woinowsky-Krieger [51] with the theory of thermoelasticity with second sound (see, e.g., [17]). The material constitutive relation is viscoelastic of the Kelvin–Voigt type.

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1.1. The PDE system

Let $T > 0$ be the final time of interest. We have

$$\begin{aligned} &u_{tt}(x,t) + \kappa u_{xxxx}(x,t) - \left[\beta + \int_0^\ell \left|u_x(x,t)^2\right| dx\right] u_{xx}(x,t) + m\theta_{xx}(x,t) + du_{xxxxt}(x,t) = 0, \\ &\theta_t(x,t) + \alpha q_x(x,t) - mu_{xxt}(x,t) = 0, \\ &\tau q_t(x,t) + q(x,t) + \alpha\theta_x(x,t) = 0, \end{aligned} \tag{1}$$

in the unknown variables $u = u(x,t) : [0,\ell] \times [0,T] \to \mathbb{R}$, $\theta = \theta(x,t) : [0,\ell] \times [0,T] \to \mathbb{R}$ and $q = q(x,t) : [0,\ell] \times [0,T] \to \mathbb{R}$. With regard to the physical meaning of the variables in play, u represents the vertical deflection of the beam from its configuration at rest, the "temperature variation" θ actually arises from an approximation of the temperature variation with respect to a reference value (see, e.g., [35]), and the only nonzero component of the heat flux is its x-component q. Throughout the paper, the subscripts x and t indicate partial derivatives. The coefficients represent:

- κ the scaled elasticity modulus,
- d the viscosity coefficient,
- α the scaled thermal diffusivity,
- τ the relaxation time,
- m the coupling coefficient depending on the material properties,

with $\kappa, d, \alpha, \tau \in \mathbb{R}^+$ and $m \in \mathbb{R} \setminus \{0\}$. In particular, the constant $\beta \in \mathbb{R}$ accounts for the axial force acting in the reference configuration: β is positive when the beam is stretched and negative when compressed.

To complete the description of the problem we consider the following initial conditions:

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \theta(x,0) = \theta_0(x), \qquad q(x,0) = q_0(x) \quad \text{in } [0,\ell], \tag{2}$$

for some given functions $u_0, u_1, \theta_0, q_0 : (0,\ell) \to \mathbb{R}$.

In order to simplify the analysis, we assume that the beam is rigidly clamped at its left end, that is:

$$u(0,t) = 0, \qquad u_x(0,t) = 0 \quad \text{in } [0,T], \tag{3}$$

and also

$$\theta_x(0,t) = 0, \qquad q(0,t) = 0 \quad \text{in } [0,T]. \tag{4}$$

For the sake of simplicity, we denote by ζ the triplet $\zeta = (u, \theta, q)$ and by $\sigma(t,\zeta)$ the shear stress at $x = \ell$, i.e.

$$\sigma(t,\zeta) := -\kappa u_{xxx}(\ell,t) + \left[\beta + \int_0^\ell \left|u_x(x,t)\right|^2 dx\right] u_x(\ell,t) - m\theta_x(\ell,t) - du_{xxxt}(\ell,t).$$

The right end of the beam is supposed to move vertically between two stops and the joint is assumed to be asymmetrical, so that $g = g_1 + g_2$, where $g_1 > 0$ and $g_2 > 0$ are, respectively, the upper and lower clearance when the system is at rest (see Fig. 1). In addition we suppose that the stops are reactive and behave as