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# Propagation profile for a non-Newtonian polytropic filtration equation with orientated convection

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A B S T R A C T

This paper is concerned with the propagation properties for a non-Newtonian polytropic filtration equation with orientated convection

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) - \vec{\beta}(x)\cdot\nabla u^q, \quad x\in\mathbb{R}^N,\ t>0,$$

where $p>1$, $m,q>0$, $N\geq 1$. Here, the orientation of the convection is specified to be either the convection with counteracting diffusion or the convection with promoting diffusion, that is $\vec{\beta}(x)\cdot(-x)\geq 0$ or $\vec{\beta}(x)\cdot x\geq 0$ respectively. By use of the super- and sub-solution method, we make complete classification of the three parameters $m$, $p$, $q$ to characterize its propagation profile of solutions.


## 1. Introduction

This paper is devoted to the following non-Newtonian polytropic filtration equation with orientated convection

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) - \vec{\beta}(x)\cdot\nabla u^q, \quad x\in\mathbb{R}^N,\ t>0, \tag{1.1}$$

where $p>1$, $m,q>0$, $N\geq 1$, and $\vec{\beta}(x)$ is a continuous and bounded vector field defined on $\mathbb{R}^N$ with the orientation either $\vec{\beta}(x)\cdot(-x)\geq 0$ or $\vec{\beta}(x)\cdot x\geq 0$ for $x\in\mathbb{R}^N$.

Equations like (1.1) arise from the study of a variety of diffusive phenomena, such as soil physics, fluid dynamics, combustion theory, reaction chemistry, see [2–4,8] and the references therein, where a more detailed physical background can be found. It is well known that the fast diffusions (such as those with $m(p-1)\leq 1$, $\vec{\beta}(x)=0$ for $x\in\mathbb{R}^N$ in (1.1), including the classical heat transformations) result in infinite

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propagation speeds of disturbances, where the solutions become positive everywhere instantaneously for any nontrivial, nonnegative initial data. Contrarily, the slow diffusive equations ($m(p-1) > 1$, $\vec{\beta}(x) = 0$ for $x \in \mathbb{R}^N$ in (1.1)) admit disturbances propagations with finite speeds, and there should be some interfaces to separate the positive and zero regions of solutions. These phenomena can be observed in many physical processes, for example, the wetting fronts separating wet and dry regions of the porous medium in the soil-moisture infiltration, the leading edge of the fluid flow in a thin viscous film with the unknown variable representing the thickness of the film [5].

During the last three decades, the topic on finite propagation properties for the non-Newtonian diffusion equations attracts a great deal of attention, see for example [7,6,11–13,16,14,9,10,15], and the references therein. Here, we mention some works on them. In 2009, T.S. Khin and Ning Su [7] studied a one-dimensional nonlinear parabolic equations of the $p$-Laplacian with convection-absorption, and established necessary and sufficient conditions for the finite propagation properties. After that, Jin, Yin and Zheng [6] considered the propagation profile for evolutionary $p$-Laplacian with convection in half space. They studied the evolution of the interface $\xi(t) = \sup\{x; u(x,t) > 0\}$, and obtained some sufficient conditions for some propagation properties, such as localization, shrinking and expanding. As for the disturbances propagation in the multi-dimensional space concerned, it is worthy of mentioning the work by Shishkov [11] in 1996, who studied a class of quasilinear doubly degenerate parabolic equations of order $2m$ in divergence form

$$(\partial/\partial t)\big(|u|^{q-1}u\big) + (-1)^m \nabla_x^m \cdot \big(\big|\nabla_x^m u\big|^{p-1} \nabla_x^m u\big) = 0$$

in $\mathbb{R}^N \times (0,T)$, where $N, m \geq 1$, $p > q > 0$. If $q, m = 1$, this reduces to the well-known evolutionary $p$-Laplacian equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big).$$

The author proved that the support of generalized solutions of the Cauchy problem varies with finite velocity under minimal regularity conditions on the initial data. Soon afterwards, Ning Su [12] studied anisotropic propagation properties for a general nonlinear parabolic equation of the form

$$\partial_t \varphi(u) - \nabla_x \cdot \mathbf{A}(\nabla_x u) + \nabla_x \cdot \mathbf{B}(u) + C(u) = 0$$

and obtained sufficient conditions for propagation properties of disturbances which propagate in a fixed direction $\alpha \in \mathbb{R}^N$, which can be thought of as some variants of that for one dimensional case.

In the present paper, we pay our attention to the finiteness of the propagation speed of disturbances in the multi-dimensional space for Eq. (1.1). For this propose, we consider the nonnegative solutions of Eq. (1.1) subject to nonnegative, continuous and compactly supported initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with $u_0(0) > 0$. We are quite interested in specified orientated convection, namely the convection with counteracting diffusion and the convection with promoting diffusion, that is $\vec{\beta}(x) \cdot (-x) \geq 0$ and $\vec{\beta}(x) \cdot x \geq 0$ for $x \in \mathbb{R}^N$ respectively, see Figs. 1 and 2.

Since that the interface of support in multi-dimensional space is more complicated than that in one-dimensional space, the corresponding studies on propagation properties of disturbances are more difficult and interesting. Firstly, necessary and sufficient conditions for finite speed of propagation of disturbances are established. Then, we study the propagation properties of supports when the disturbances propagate with finite speeds, such as localization and expanding. And one will see that the interaction between diffusion and convection contributes substantial influences on the behavior of solutions. In detail, for