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Young's inequality for convolution and its applications in convex- and set-valued analysis

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ABSTRACT

A version of Young's inequality for convolution is introduced and employed to some topics in convex- and set-valued analysis. The following problems are considered: uniform equivalence of metrics for convex subsets of the Euclidean space, the regularity of set-valued mappings and the continuity of the Funk–Radon transform. Also an isoperimetric inequality based on Young's inequality is introduced.

1. Introduction

The goal of this paper is to establish a link between Young's inequality for convolution and some problems in convex- and set-valued analysis. Young's inequality for convolution $I(x) = \int_{\mathbb{R}^n} K(y)f(x-y)\,dy$ asserts that

$$\|I\|_{L_q(\mathbb{R}^n)} \leq \|K\|_{L_r(\mathbb{R}^n)} \cdot \|f\|_{L_p(\mathbb{R}^n)} \tag{1}$$

provided $K : \mathbb{R}^n \mapsto \mathbb{R}$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ are mappings such that $K \in L_r(\mathbb{R}^n)$, $f \in L_p(\mathbb{R}^n)$ and $1 \leq p \leq q \leq \infty$, $1 - \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This inequality attracted the attention of many authors – for instance, inequality (1) was generalized into an inequality for mappings defined on an abstract group [11]. Other ideas, such as the reverse inequality

$$\exists C \geq 0 \quad \|I\|_{L_q(\mathbb{R}^n)} \geq C \cdot \|K\|_{L_r(\mathbb{R}^n)} \cdot \|f\|_{L_p(\mathbb{R}^n)},$$

have also been introduced in [4]. Our aim is to derive another version of Young's inequality and describe its applications. We begin with the question to derive an analogue of (1) under weaker assumptions. Mappings

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$K : \mathbb{R}^n \mapsto \mathbb{R}$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ are supposed to be at most locally integrable in L_r and in L_p, respectively.

An estimate similar to Young's inequality is introduced. This estimate is employed to investigate a few topics appearing in convex- and set-valued analysis:

1. The problem of *uniform equivalence of metrics* for convex (or star-shaped) subsets of the Euclidean space $\mathbb{R}^n$.
2. *Uniform continuity of multifunctions.* Let F be a multifunction from a metric space to convex compact subsets of $\mathbb{R}^n$. If F is uniformly continuous in a metric d_1 with modulus of continuity ω_1, then Young's inequality is a tool to verify the continuity of F in a metric d_2 and calculate the modulus of continuity of F in metric d_2.
3. *Intersection bodies* and the Funk–Radon transform – a result on the continuity of the Funk–Radon transform.
4. *Isoperimetric inequalities.* An isoperimetric inequality following from Young's inequality is established.

These problems have been studied in the literature and our aim is to develop some of known results using a version of Young's inequality for convolution. Now let us briefly discuss topics introduced above.

1.1. Uniform equivalence of metrics for convex subsets of the Euclidean space $\mathbb{R}^n$

Apart from the well known Hausdorff distance there are many other metrics for convex subsets of $\mathbb{R}^n$ such as the Demyanov metric, the integral metric, the radial metric or the symmetric difference metric. The relationship between these metrics was studied in [3,8,10,14,17,18,20,25,26] and we would like to extend some of known results.

There are two types of equivalence of metrics: the *topological* and the *uniform* equivalence. These concepts are defined as follows: suppose there are a nonempty set $\mathcal{A}$ and metrics d_1, d_2 on $\mathcal{A}$. Then metrics d_1, d_2 are

1. *topologically equivalent* if every sequence in $\mathcal{A}$ convergent in d_1 is also convergent in d_2 and every sequence in $\mathcal{A}$ convergent in d_2 is also convergent in d_1;
2. *uniformly equivalent* on $\mathcal{A}$ if the identity mappings $id_1 : (\mathcal{A}, d_1) \mapsto (\mathcal{A}, d_2)$ and $id_2 : (\mathcal{A}, d_2) \mapsto (\mathcal{A}, d_1)$ are uniformly continuous.

Let us recall some of published results on the equivalence of metrics. Florian [10] compared the Hausdorff metric d_H and the integral metric δ_1. It was shown that these two metrics are topologically equivalent on the space $\mathcal{X} = \{A \subset \mathbb{R}^n : A \neq \emptyset \text{ is convex and compact}\}$. However, metrics d_H and δ_1 are not uniformly equivalent on $\mathcal{X}$, because there are sequences $\{K_j\}$ and $\{H_j\}$ of elements of $\mathcal{X}$ such that $\delta_1(K_j, H_j) \to 0$ as $j \to +\infty$, but $d_H(K_j, H_j) \nrightarrow 0$.

Vitale [26] compared the Hausdorff metric d_H and the integral metric δ_p. These metrics are not uniformly equivalent on the space $\mathcal{X}$, but they are uniformly equivalent on a subspace $\mathcal{Y}$ of $\mathcal{X}$, i.e.

$$\begin{aligned} &\exists\big(C > 0;\ \alpha \in (0,1)\big)\ \forall(A_1, A_2 \in \mathcal{Y} \subset \mathcal{X}) : \\ &d_H(A_1, A_2) \le C \cdot \big(\delta_p(A_1, A_2)\big)^{\alpha} \quad \text{and} \quad \delta_p(A_1, A_2) \le C \cdot d_H(A_1, A_2). \end{aligned} \tag{2}$$

In comparison to these results, in [8,14,20,25] other pairs of metrics were studied and similar results were established. For instance, in [20] the Hausdorff metric d_H and the Demyanov metric d_D were compared:

$$\exists C > 0\ \forall K_1, K_2 \in \mathcal{Y}' \subset \mathcal{X} : \quad d_D(K_1, K_2) \le C \cdot \big(d_H(K_1, K_2)\big)^{1/2}. \tag{3}$$