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Wave diffraction by wedges having arbitrary aperture angle

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A R T I C L E I N F O

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A B S T R A C T

The problem of plane wave diffraction by a wedge sector having arbitrary aperture angle has a very long and interesting research background. In fact, we may recognize significant research on this topic for more than one century. Despite this fact, up to now no clear unified approach was implemented to treat such a problem from a rigourous mathematical way and in a consequent appropriate Sobolev space setting. In the present paper, we are considering the corresponding boundary value problems for the Helmholtz equation, with complex wave number, admitting combinations of Dirichlet and Neumann boundary conditions. The main ideas are based on a convenient combination of potential representation formulas associated with (weighted) Mellin pseudo-differential operators in appropriate Sobolev spaces, and a detailed Fredholm analysis. Thus, we prove that the problems have unique solutions (with continuous dependence on the data), which are represented by the single and double layer potentials, where the densities are solutions of derived pseudo-differential equations on the half-line.

1. Introduction

The problem of plane wave diffraction by wedge sectors counts already more than one century of research. Indeed, we may identify the classical works of Sommerfeld [67] and Poincaré [60] as the first ones where this type of problem was significantly tackled. There, the solution of the Helmholtz equation in an infinite wedge sector with Dirichlet and Neumann boundary conditions was studied by using the Sommerfeld integrals and separation of variables, respectively. Anyway, previous partial results can also be identified. This is the case of Macdonald [39] who already gave in 1895 a representation of the first and second Green's functions (i.e., electrostatic and velocity potentials) of the potential equation for a wedge of an arbitrary aperture angle. In fact, this was first considered only for angles of the form π/m, where m is a positive integer, and later (cf. [40]) Macdonald was able to obtain formulas for the two Green's functions of the Helmholtz equation

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for wedges with any aperture angle. However, Macdonald's method is not easy to follow when involving somehow conventional formalisms of nineteenth century.

Carslaw did also relevant work on the construction of appropriate potentials, by using the Sommerfeld's method, first for some wedges of particular aperture angles and then for arbitrary ones (cf. [3–5]).

In the last decades the mathematical analysis of wave diffraction problems by wedge configurations has been receiving increased attention. Consequently, we can identify a significant number of publications where such analysis was taken for particular cases of wedge amplitudes and/or boundary conditions (cf., e.g., [2,7–13,18,21,22,33–35,42,48,44–46,49,50,55,56,58,61,59,72]). However, none of these listed papers contain final solvability results for the general problems in a rigourous mathematical Sobolev space setting as is done in the present paper.

It is clear that one of the main difficulties in such analysis arises from the geometry of the domain in consideration. For some regions, the direct method of layer potentials works very well, allowing the well-posedness of the problems in appropriate Sobolev spaces and, in some cases, closed-form solutions. For smooth domains the list of publications is quite huge. Anyway, we would like to refer here to some corresponding excellent works which present a somehow rather complete account of the theory in smooth domains, as is the case of the books by Colton and Kress [14], Courant and Hilbert [17], Hsiao and Wendland [26], Kress [37], McLean [41], and Taylor [69]. Moreover, among the non-smooth domains, general theories for Lipschitz domains are also available and can be tested in concrete corresponding boundary value problems. Related to this, we would like to refer the works of Costabel [15], Costabel and Stephan [16], Jerrison and Kenig [27–29], Kohr, Pintea and Wendland [31], Mitrea and Mitrea [51,52], Mitrea and Taylor [53,54], and Verchota [71].

We may say that the recent developments in problems of wave diffraction by non-smooth regions were certainly inspired by also somehow recent significant general results for boundary value problems in non-smooth domains. As representatives of the latter ones, we may also cite here the monographs [19,25,57,64], as well as the pertinent work [36]. Here, Kondrat'ev's method is mainly based on the Mellin transform, already allowing information on the smoothness and asymptotic expansion of the solutions at the edges of the boundary angles.

The relevant work of Komech, Merzon and their collaborators [32–35] must also be referred, where the so-called method of complex characteristics for elliptic equations in nonconvex angles is used. Typically, the crucial part of the method is played by the connection equation on the Riemann surface of complex characteristics of the given elliptic operator. Also, the limiting amplitude principle in the two-dimensional scattering of an incident plane harmonic wave by a wedge has recently been successfully applied, cf. [13,49,56].

In [18], the problem of wave diffraction by impenetrable wedges having arbitrary aperture angle was studied by means of the Wiener–Hopf method. This positively answered the important issue (that had been open for a long period) on the possibility of applying the Wiener–Hopf technique to this more complex geometrical problem of having wedges with arbitrary angles. However, no concern with the space setting was there presented. As a very significant result, it was obtained that the diffraction by an impenetrable wedge always reduces to a standard Wiener–Hopf factorization. For given impedance boundary conditions, explicit factorizations were derived which lead to consequent closed-form solutions.

The series of results obtained by Meister, Speck and their collaborators (cf., e.g., [10–12,21,22,42,48, 44–46,58]) constitute a systematic approach to a rigourous mathematical analysis of plane wave diffraction by wedges. They obtained important conclusions on both the well-posedness of the problems and consequent closed-form solutions, in appropriate Sobolev spaces, for a large number of particular cases of aperture angles and different types of boundary conditions. In particular, in [21,22], the authors obtained the well-posedness for the so-called rational angles of the form $\pi m/n$, where m and n are natural numbers. This was done by using symmetry properties within certain Riemann surfaces. In addition, this was a somehow natural development of the previous work [58] where, by using also symmetry arguments and Sommerfeld potentials (resulting from special Sommerfeld problems which are explicitly solvable), the well-posedness and explicit