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Hypergeometric transformation formulas of degrees 3, 7, 11 and 23

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ABSTRACT

The theory of theta functions is used to derive hypergeometric transformation formulas of degrees 3, 7, 11 and 23. As a consequence of the theory that is developed, some new series for $1/\pi$ are obtained that are similar to a class investigated by Ramanujan.

1. Introduction

One of the fundamental results in the theory of elliptic integrals is the quadratic transformation formula

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\left(\frac{1-x}{1+x}\right)^2\right) = (1+x)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right). \tag{1.1}$$

Its application to the arithmetic–geometric mean (AGM) iteration is described in [1,6,11] and [15]. The formula (1.1) may be iterated to produce the quartic transformation formula

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\left(\frac{1-x}{1+x}\right)^4\right) = (1+x)^2\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^4\right). \tag{1.2}$$

Other examples, analogous to (1.1) and (1.2), are the following quadratic and cubic transformation formulas given by Ramanujan in his second notebook [26, pp. 258, 260]:

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$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-\left(\frac{1-x}{1+3x}\right)^2\right)=\sqrt{1+3x}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;x^2\right) \tag{1.3}$$

and

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{1-x}{1+2x}\right)^3\right)=(1+2x)\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x^3\right). \tag{1.4}$$

Proofs of (1.3) and (1.4) were given in [2, pp. 97, 146] and [4]. The transformation formulas (1.2), (1.3) and (1.4) have been used to produce iterations that are analogues of the AGM in [7–10]. Uniform proofs of all of (1.1)–(1.4) have since been given in [15] and [23].

In contrast to the old results (1.1)–(1.4), hypergeometric transformation formulas of degree ≥ 5 have been discovered only recently. Hypergeometric transformation formulas of degrees 5, 6 and 7 were given by R. Maier [23]. Other hypergeometric transformation formulas of degree 5 appear in [16, Section 4] and [20, Theorem 6.5], a different formula of degree 7 is given in [20, Theorem 4.5], and a hypergeometric transformation formula of degree 13 is given in [19, Theorem 5.10].

The goal of this work is to produce a family of hypergeometric transformation formulas of degrees 3, 7, 11 and 23. We share Maier's view [23] that the transformation formulas are not most simply expressed in terms of ${}_2F_1$. However, whereas Maier expressed transformation formulas of higher degrees in terms of Heun functions, we find relatively simple formulas in terms of ${}_3F_2$ functions. Clausen's formula could be used to convert the results into ${}_2F_1$ hypergeometric functions, if required. As an example, the result in the case of degree 7 that we shall obtain is

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}^2\binom{2j}{n}\binom{j+n}{j}\right\}\left(\frac{t}{1+13t+49t^2}\right)^n \\
&\quad=\sqrt{\frac{1+13t+49t^2}{1+245t+2401t^2}}\,{}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;\frac{1728t}{(1+13t+49t^2)(1+245t+2401t^2)^3}\right) \\
&\quad=\sqrt{\frac{1+13t+49t^2}{1+5t+t^2}}\,{}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;\frac{1728t^7}{(1+13t+49t^2)(1+5t+t^2)^3}\right),
\end{aligned} \tag{1.5}$$

valid in a neighborhood of $t=0$. The second equality in (1.5) is a hypergeometric transformation formula that is different from the transformation formulas of degree 7 in [20, Theorem 4.5] and [23, Proposition 6.4] that are formulated in terms of hypergeometric and Heun functions, respectively. We will prove (1.5) as part of Theorem 5.1, which also contains analogous transformation formulas of degrees 3, 11 and 23.

The method of proof makes use of analogues of Jacobi's sums of four and eight squares identities, as well as the corresponding analogues of the Glaisher–Ramanujan sums of 12, 16 and 20 squares identities. We also recover some series for $1/\pi$ of Ramanujan and exhibit some series for levels 11 and 23 that are new.

This work is organized as follows. Definitions and notation are set out in the next section. The primary functions to be studied are the weight 2 modular form z_p and the associated modular function x_p. Here, and from now on, p will always denote one of the integers 3, 7, 11 or 23, called the level. Sometimes p will be referred to as the degree. In Section 3, the Dedekind eta function, the modular j invariant and certain Eisenstein series are expressed in terms of x_p and z_p. Proofs of these results rely on the analogues of the sum of 4, 8, 12, 16 and 20 squares identities. Differential equations for z_p in terms of x_p are established in Section 4. The main results occur in Section 5. In Theorem 5.1 each function z_p is expressed as a power series in x_p, and also in two ways in terms of hypergeometric functions of x_p. This gives rise to hypergeometric transformation formulas of degree p. We describe how the results in the case $p=3$ can be used to prove Ramanujan's formula (1.4). The case $p=7$ of Theorem 5.1 gives the identity (1.5). In Section 6 we produce some series for $1/\pi$ that include Ramanujan's cubic examples [25], recent septic examples [17], as well as some series for levels 11 and 23 that are new.