J. Math. Anal. Appl. 421 (2015) 1417–1427

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some results about operator perturbation of fusion frames in Hilbert spaces ☆

Xue-Bin Li [a], Shou-Zhi Yang [a,*], Yu-Can Zhu [b]

[a] *Department of Mathematics, Shantou University, Shantou, 515063 Guangdong, PR China*
[b] *Department of Mathematics, Fuzhou University, Fuzhou, 350108 Fujian, PR China*

A R T I C L E   I N F O

Article history:
Received 22 January 2014
Available online 1 August 2014
Submitted by R. Curto

Keywords:
Fusion frames
Fusion frame sequences
Operator perturbation
Gap
Hilbert spaces

A B S T R A C T

The stabilities of fusion frames under operator perturbation (simple named operator perturbation of the fusion frames), is the study object of this paper. We provide a way as follows for studying the operator perturbation of fusion frames. Firstly, we consider the operator perturbation of fusion frame sequences by the gap between two closed subspaces and discuss the relationships between the operator perturbation of fusion frames and the operator perturbation of fusion frame sequences. Lastly, we transform the operator perturbation of fusion frames into the operator perturbation of fusion frame sequences based on these relationships. Our results obtained this way generalize the remarkable results which have been obtained by Casazza, Kutyniok, Asgari, Gavruta and Zhu.

1. Introduction

In 1952, frames were first introduced by Duffin and Schaeffer [9]. In recent 30 years, it was widely studied and applied in sigma–delta quantization [3], filter bank theory [4], signal and image processing [5] and so on. Fusion frames which are the generalization of frames, were studied recently by Casazza, Fornasier and so on. Now many important results about fusion frames have been obtained by many authors (see [1,2,6,7,10,13–15]).

We note that H and K are two Hilbert spaces, $L(H,K)$ is the space of all bounded linear operators from H to K, especially, $L(H)$ is the space of all bounded linear operators on H. $\{W_j\}_{j=1}^{\infty}$ is a family of closed subspaces of H and $\{v_j\}_{j=1}^{\infty}$ is a family of positive weights, i.e. $v_j > 0$. π_{W_j} is an orthogonal projection from H onto W_j for every $j \in Z^+$, where Z^+ is a set of positive integers. Let $I \subset Z^+$. The space $l^2(\{W_j\}_{j\in I})$ is defined by

☆ This work was supported by the National Natural Science Foundation of China (Grant No. 11071152), the Natural Science Foundation of Guangdong Province (Grant No. S2011010004511), and the Natural Science Foundation of Fujian Province, China (Grant No. 2012J01005).

* Corresponding author.
E-mail address: szyang@stu.edu.cn (S.-Z. Yang).

http://dx.doi.org/10.1016/j.jmaa.2014.07.068
0022-247X/© 2014 Elsevier Inc. All rights reserved.

$$l^2\big(\{W_j\}_{j\in I}\big) = \left\{\{f_j\}_{j\in I}\middle| f_j \in W_j \text{ and } \sum_{j\in I}\|f_j\|^2 < \infty\right\},$$

with inner product given by

$$\big\langle \{f_j\}_{j\in I}, \{g_j\}_{j\in I}\big\rangle = \sum_{j\in I}\langle f_j, g_j\rangle.$$

Let $T \in L(H, K)$ be a closed range operator and n be a positive integer. Note that $\mathcal{W}_n = l^2(\{W_j\}_{j=1}^n) = \{\{f_j\}_{j=1}^n | f_j \in W_j \text{ and } 1 \leq j \leq n\}$ and $l^2(\{H\}_{j=1}^n) = \{\{f_j\}_{j=1}^n | f_j \in H \text{ and } 1 \leq j \leq n\}$, we let $\pi_{\mathcal{W}_n}$ be an orthogonal projection from $l^2(\{H\}_{j=1}^n)$ onto $l^2(\{W_j\}_{j=1}^n)$, and let

$$\mathcal{U}_{n,T} = \big\{\big\{v_j T^* \pi_{\overline{TW_j}} f\big\}_{j=1}^n \big| f \in K\big\}.$$

By some straightforward computations, we obtain

$$\pi_{\mathcal{W}_n}\big(\{f_j\}_{j=1}^n\big) = \{\pi_{W_j} f_j\}_{j=1}^n \tag{1.1}$$

for any $\{f_j\}_{j=1}^n \in l^2(\{H\}_{j=1}^n)$.

Definition 1.1. Let $\{W_j\}_{j=1}^\infty$ be a family of closed subspaces of H, $\{v_j\}_{j=1}^\infty$ be a family of positive weights, i.e. $v_j > 0$ for $j = 1, 2, \cdots$. If there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{j=1}^\infty v_j^2 \big\|\pi_{W_j}(f)\big\|^2 \leq B\|f\|^2$$

for all $f \in H$, then we call $\{(W_j, v_j)\}_{j=1}^\infty$ a fusion frame for H with bounds A and B.

Definition 1.2. A family $\{(W_j, v_j)\}_{j=1}^\infty$ is a fusion frame sequence for H, if $\{(W_j, v_j)\}_{j=1}^\infty$ is a fusion frame for $\mathcal{W}$ with bounds A and B, where

$$\mathcal{W} = \overline{span}\{W_j\}_{j=1}^\infty = \overline{\left\{\sum_{j\in I} f_j \middle| f_j \in W_j, I \subseteq Z^+ \text{ is finite set}\right\}}.$$

The positive real constants A and B are called lower and upper bounds of the fusion frame sequence.

Definition 1.3. Let W and V be two closed subspaces of H. We define the gap between W and V by

$$\delta(W, V) = \sup_{x\in W, \|x\|=1} \operatorname{dist}(x, V) = \sup_{x\in W, \|x\|=1} \inf_{y\in V} \|x - y\|.$$

The thing to note here is that $0 \leq \delta(W, V) \leq 1$. Especially, if $W \subset V$ then $\delta(W, V) = 0$; if W and V are two orthogonal closed subspaces of H then $\delta(W, V) = 1$. Some results about the gap between two subspaces see [11].

Definition 1.4. Suppose that H, K are two Hilbert spaces, and $T : H \to K$ is a bounded linear operator. Let

$$\gamma(T) = \inf\big\{\|Tx\| : x \in N_T^\perp, \|x\| = 1\big\},$$

then we call $\gamma(T)$ the minimum modulus of operator T.