J. Math. Anal. Appl. 421 (2015) 1469–1478

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some applications of variation of Loewner chains in several complex variables

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ARTICLE INFO

Article history:
Received 21 March 2014
Available online 8 August 2014
Submitted by V. Andrievskii

Keywords:
Univalent mapping
Loewner chain
Loewner–Kufarev PDE
Runge domain
Embedding problem
Variational method

ABSTRACT

In a recent paper, Bracci, Graham, Hamada, Kohr developed a new method to construct Loewner chains, by considering variations of certain Loewner chains. We use this method first, to prove a topological property of the class $\mathcal{S}^0$ of mappings with parametric representation on the Euclidean unit ball $\mathbb{B}^n$, which for $n \geq 2$ immediately implies the density of the automorphisms of $\mathbb{C}^n$ that, restricted to $\mathbb{B}^n$, have parametric representation, and second, to prove that every normalized univalent mapping on $\mathbb{B}^n$ whose image is Runge and which is C^1 up to the boundary embeds into a normalized Loewner chain with range $\mathbb{C}^n$.

1. Introduction

The theory of Loewner chains in one complex variable, introduced by Charles Loewner in 1923 [17], has been proved to be a very powerful tool in various topics in mathematics. For a view of the development of this theory, with many references, one can consult [1].

The theory of Loewner chains was extended to several complex variables by Pfaltzgraff [18], Poreda [21,22], Graham, Hamada, Kohr, Kohr [10,11,15,16], and many others. This theory was developed also in more general settings, like e.g. Banach spaces [13] and complex manifolds [4,7].

Some striking differences between one complex variable and several complex variables, regarding e.g. Runge and Stein domains or Andersén–Lempert theory, are reflected also in the theory of Loewner chains – see [6,5].

A very active topic in this theory is the study of extreme and support points of certain families of holomorphic mappings – see e.g. [8,12,14,24,25]. In [8] a new variational method was developed to study extreme and support points of the class of mappings with parametric representation on the Euclidean unit

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http://dx.doi.org/10.1016/j.jmaa.2014.07.084
0022-247X/

ball. Also, in [24], a related variational method was used very efficient to obtain a Pontryagin maximum principle for the Loewner differential equation and to study support points of the same class.

In this paper we give some new applications of the variational method from [8], but in a different context.

In the following we recall some definitions, notations and results from [6,8], in order to state and prove our results.

We consider the Euclidean inner product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|\cdot\|$ on $\mathbb{C}^n$. $\mathbb{B}^n$ denotes the Euclidean unit ball in $\mathbb{C}^n$.

Let

$$\mathcal{H}_0 := \left\{f : \mathbb{B}^n \to \mathbb{C}^n \;\middle|\; f \text{ is holomorphic and } f(0) = 0,\ df_0 = I\right\}$$

and

$$\mathcal{S} := \{f \in \mathcal{H}_0 \mid f \text{ is univalent}\}.$$

Definition 1.1. A family $(f_t)_{t\geq 0}$ of mappings is called a normalized subordination chain if $\{e^{-t}f_t\}_{t\geq 0}$ is a family in $\mathcal{H}_0$ and, for every $0 \leq s \leq t$, there exists $\varphi_{s,t} : \mathbb{B}^n \to \mathbb{B}^n$ holomorphic such that $f_s = f_t \circ \varphi_{s,t}$.

A normalized subordination chain $(f_t)_{t\geq 0}$ is called a normalized Loewner chain if $\{e^{-t}f_t\}_{t\geq 0}$ is a family in $\mathcal{S}$, and if, in addition, $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family, then $(f_t)_{t\geq 0}$ is called a normal Loewner chain.

For every normalized subordination chain $(f_t)_{t\geq 0}$ we denote

$$R(f_t) := \bigcup_{t\geq 0} f_t(\mathbb{B}^n).$$

This set is called the (Loewner) range of $(f_t)_{t\geq 0}$.

We say that a mapping $f \in \mathcal{S}$ embeds into a normalized Loewner chain $(f_t)_{t\geq 0}$ if $f_0 = f$.

Let

$$\mathcal{S}^0 := \left\{f \in \mathcal{S} \;\middle|\; f \text{ embeds into a normal Loewner chain } (f_t)_{t\geq 0}\right\},$$
$$\mathcal{S}^1 := \left\{f \in \mathcal{S} \;\middle|\; f \text{ embeds into a normalized Loewner chain } (f_t)_{t\geq 0} \text{ with } R(f_t) = \mathbb{C}^n\right\}$$

and

$$\mathcal{S}_R := \left\{f \in \mathcal{S} \;\middle|\; f(\mathbb{B}^n) \text{ is Runge}\right\}.$$

For the definition and basic properties of the Runge domains, one can consult [23, Chapter VI, Section 1.4].

The class $\mathcal{S}^0$ is known as the class of mappings with parametric representation on $\mathbb{B}^n$ (see [15, Section 8.3]) and it is a compact set in $\mathcal{H}_0$ with respect to the compact-open topology (see [15, Corollary 8.3.11]).

Let

$$\mathcal{M} := \left\{h \in \mathcal{H}_0(\mathbb{B}^n) \;\middle|\; \Re\langle h(z), z\rangle \geq 0 \text{ for all } z \in \mathbb{B}^n\right\}.$$

Various applications of this family in the theory of univalent mappings on $\mathbb{B}^n$ may be found in [15] and [27], and the references therein.

Definition 1.2. A mapping $G : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is called a Herglotz vector field if $G(z,\cdot)$ is measurable, for every $z \in \mathbb{B}^n$, and $G(\cdot,t) \in \mathcal{M}$, for a.e. $t \in [0,\infty)$.