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A piecewise quadratic maximum entropy method for the statistical study of chaos

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ABSTRACT

Let the Frobenius–Perron operator $P_S : L^1(0,1) \to L^1(0,1)$, related to a nonsingular transformation $S : [0,1] \to [0,1]$, have an invariant density f^*. We propose a piecewise quadratic maximum entropy method for the numerical approximation of f^*. The role of the partition of unity property of the quadratic functions for the numerical recovery of f^* has been depicted. The proposed algorithm overcomes the ill-conditioning shortage of the traditional maximum entropy method which only employs polynomials, so that any number of moments can be used to increase the accuracy of the computed invariant density. The convergence rate of the method is shown to be of order 3. Numerical results are presented to justify the theoretical analysis of the method.

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1. Introduction

Moment problems can be found in spectra estimation, geophysics, radio astronomy, sonar and radar (see [12,13] and references therein). They also arise in theoretical physics such as quantum spin systems, Ising models, and the divergent series obtained from Stieltjes transformations [15]. The moment problem has appeared in pure mathematics since Hausdorff [10].

In 1866, L. Boltzmann used the concept of entropy for the kinetic theory of gases. The second law of thermodynamics gives that, in an isolated system, the entropy never decreases. Jaynes introduced the principle of maximum entropy in 1957. In his paper [11] he formulated the maximum entropy problem to recover a least biased density among all densities satisfying a finite number of given constraints. This concept has been widely used in different fields of science and engineering [15,17].

The author of [4] developed a method for solving the fixed point equation of Frobenius–Perron operators based on Jayens' maximum entropy principle and it was extended to find the Lyapunov exponents of chaotic maps in [6]. A discrete version of the maximum entropy method for computing invariant densities of the

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Frobenius–Perron operator and the Lyapunov exponent using Boltzmann entropy functional can be found in [1], whereas its continuous version was used in [7] based on orthogonal polynomials.

The use of polynomials in the continuous version of maximum entropy method for the recovery of an invariant density leads to solving systems of nonlinear equations. The difficulty arises in solving such systems when they are in ill-condition. The difficulty can be resolved by taking piecewise polynomial functions instead of taking polynomials over the whole domain.

Piecewise constant polynomials to approximate the Frobenius–Perron operator in the maximum entropy method were used in [8], which was the first publication on the maximum entropy method based on piecewise polynomials to recover densities. The authors of [9] considered piecewise linear functions for the approximation of an invariant density of the Frobenius–Perron operator.

In this paper, we use piecewise quadratic functions for solving the maximum entropy problem to numerically recover an invariant density of the Frobenius–Perron operator associated with a nonsingular transformation from $[0, 1]$ to itself. The use of piecewise quadratic functions as the moment functions gives a system of nonlinear equations. The Jacobian matrix of the system is five-diagonal, nonsingular and positive definite. Newton's method can be implemented to solve the nonlinear system.

In Section 2, we give the definition of and brief introduction to maximum entropy and the Frobenius–Perron operator. We discuss the Gibbs inequality and the solution of the maximum entropy problem under finite constraints. We give the formulation of piecewise quadratic functions, some of their features, and their use in the maximum entropy problem in Section 3. In Section 4, we outline a convergence analysis of the proposed method and give some error estimates for the sake of practical implementation. In Section 5, we discuss the numerical results and compare the errors from the new method with two existing methods, and present them in tabular forms. We give the conclusion in Section 6.

2. Maximum entropy and the Frobenius–Perron operator

2.1. Maximum entropy

We start with the Boltzmann entropy. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. If $f \geq 0$ and $\eta(f) \in L^1(X, \mu)$ then the entropy of f is defined by

$$H(f) = \int_X \eta\big(f(x)\big)\mu(dx), \tag{1}$$

where

$$\eta(u) = \begin{cases} -u \ln u & \text{if } u > 0, \\ 0 & \text{if } u = 0. \end{cases} \tag{2}$$

Remark 2.1. When $\mu(X) < \infty$, the integral (1) is well defined for every $f \geq 0$.

Some properties of the Boltzmann entropy $H(f)$ described below can be found in [3,14].

i) $H(f)$ is either finite or $-\infty$.

ii) H is a proper, upper semicontinuous, concave functional and strictly concave on the set,

$$\big\{f \in L^1(X, \mu), f \geq 0 : H(f) > -\infty\big\}.$$

iii) The upper level sets $\{f \in L^1(X, \mu) : f \geq 0, H(f) \geq \alpha\}$ for all $\alpha \in \mathbb{R}$ are weakly compact.

A function $f \in L^1(X, \mu)$ is called a density if $f \geq 0$ and $\|f\| \equiv \int_X f(x)\mu(dx) = 1$. The set of all densities is denoted by D.